THE APPLICATION OF INTEGRAL METHOD IN THE SOLUTION OF TRANSIENT HEAT CONDUCTION PROBLEMS IN SOLIDS WITH HEAT GENERATION

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Rezunat. Metoda integrală a fost aplicată pentru a rezolva problemele de transfer de căldură prin conducție în regim nestaționar care implica surse de căldură. Pentru a prezenta aplicația am examinat problema unei regiuni semi-infinite cu surse de căldură variabile în timp.
Cuvinte cheie: conducție in regim nestaționar, metoda integrală, surse de căldură.

Abstract. The integral method has been applied to solve transient heat conduction problems involving heat generation within the solids. To illustrate the application we examinate the problem of semi-infinite region with time dependent heat generation.
Keywords: transient conduction, integral method, heat generation

1. INTRODUCTION

Consider that a semi-infinite solid ($x \geq 0$) is initially at a constant temperature T_i. For times $\tau>0$ heat is generated within the solid at a rate of q_v W/m^3 while the boundary at $x = 0$ is kept at a constant temperature T_p. The boundary value problem of transient heat conduction is given as

$$\frac{\partial^2 T}{\partial x^2} + \frac{q_v}{\lambda} = \frac{1}{a} \cdot \frac{\partial T}{\partial \tau} \text{ in } 0 \leq x < \infty,\ \tau > 0 \quad (1a)$$

$$T = T_0 \text{ at } x = 0,\ \tau > 0 \quad (1b)$$

$$T = T_i \text{ in } 0 \leq x < \infty,\ \tau = 0 \quad (1c)$$

where $T = T(x,\tau)$, [1].

Using the thermal layer which in the heat conduction problem is defined as the distance from the origin beyond which the initial temperature distribution within the region remains unaffected by the applied boundary condition, and hence there is not heat flow in the region beyond $\delta(\tau)$, [2].

Integrate eq. 1 from $x = 0$ to $x = \delta(\tau)$

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(\tau)} - \left.\frac{\partial T}{\partial x}\right|_{x=0} + \int_0^{\delta(\tau)} \frac{q_v}{\lambda} = \frac{1}{a} \cdot \left[\frac{\partial}{\partial \tau} \int_0^{\delta(\tau)} T \mathrm{d}x - T\big|_{\delta(\tau)} \cdot \frac{\mathrm{d}\delta(\tau)}{\mathrm{d}\tau} \right] \quad (2)$$

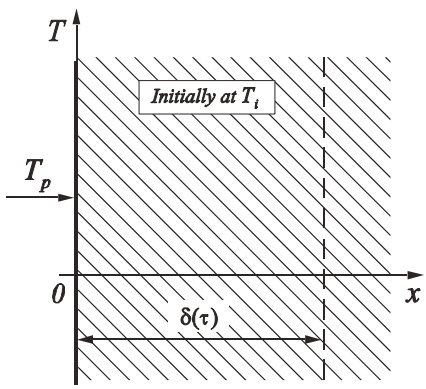

Fig. 1 Semi - infinite region with heat generation source.

The various terms in eq. 2 are:

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta(\tau)} = 0 \quad (3a)$$

$$\int_0^{\delta(\tau)} \frac{q_v(\tau)}{a} \mathrm{d}x = \frac{\delta(\tau)}{a} \cdot q_v(\tau) \quad (3b)$$

$$T\big|_{\delta(\tau)} = T_i + + \frac{1}{\rho c_p} \int_0^{\tau} q_v(\tau) \cdot d\tau \quad (3c)$$

The rezult 3a is obtained by using the definition of thermal layer $\delta(\tau)$ and 3c is obtained from the differential equation of heat conduction by

evaluating eq. 1 at $x = \delta(\tau)$ and then integrating it from $\tau = 0$ to τ, [3].
Substituting eq. 3 into eq. 2

$$-a\left.\frac{\partial T}{\partial x}\right|_{x=0} + \frac{1}{\rho c_p}\cdot\delta\cdot q_v(\tau) = \frac{\partial}{\partial\tau}\int_0^{\delta} T\mathrm{d}x -$$

$$-T_i\frac{\partial\delta}{\partial\tau} - \frac{1}{\rho c_p}\cdot\frac{\partial\delta}{\partial\tau}\cdot\int_0^{\tau} q_v(\tau)d\tau \qquad (4)$$

Defining

$$\Gamma = \int_0^{\delta} T\cdot\mathrm{d}x \qquad (5a)$$

$$Q = \int_0^{\tau} q_v(\tau)\cdot d\tau \qquad (5b)$$

Eq. 4 is writen in the form

$$-a\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{\partial}{\partial\tau}\left[\Gamma - \frac{Q\cdot\delta}{\rho c_p} - T_i\cdot\delta\right] \qquad (6)$$

Equation 6 is the heat – balance integral for the problem under consideration.

2. POLYNOMIAL APPROXIMATION

It assume a cubic degree polynomial approximation of temperature profile in the form, [2]:

$$T = ax^3 + bx^2 + cx + d \text{ in } 0 \le x \le \delta(\tau), \qquad (7)$$

The four unknown coefficients a, b, c and d are determined from the following four conditions:

$$\left.T\right|_{x=0} = T_p \qquad (8a)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=\delta} = 0 \qquad (8b)$$

$$\left.T\right|_{x=\delta} = T_i + \frac{Q}{\rho c_p} \qquad (8c)$$

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x=\delta} = 0 \qquad (8d)$$

Then the temperature profile becomes in the form

$$T = T_0 + \left[1 - \left(1 - \frac{x}{\delta}\right)^3\right]\cdot F \text{ in } 0 \le x \le \delta(\tau) \qquad (9)$$

where

$$F = \left(T_i - T_p\right) + \frac{Q}{\rho c_p} \qquad (10)$$

Substituting eq. 9 into the heat balance integral 6 obtain the following differential equation

$$12aF^2 = (F\delta)\frac{d(F\cdot\delta)}{d\tau} \text{ with } \delta = 0 \text{ for } \tau = 0 \qquad (11)$$

Solution of eq. 11 gives the following relation for the thermal layer thickness:

$$\delta^2 = 24a\frac{\int_0^{\tau} F^2 d\tau}{F^2} \qquad (12)$$

3. CONCLUSION

Equation 9 together with the value of δ evaluated from eq. 12 gives the temperature distribution profile in solid.

For no heat generation within the solid, eq. 9 becomes

$$\frac{T - T_i}{T_p - T_i} = \left(1 - \frac{x}{\delta}\right)^3 \text{ in } 0 \le x \le \delta(\tau) \qquad (13)$$

and the relation 12 for the thermal layer thickness reduces to

$$\delta = \sqrt{24a\tau} \qquad (14)$$

In the above example we considered the case in which heat generation was a function of time.

4. REFERENCES

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