MAXIMUM ATTAINABLE PERFORMANCE OF STIRLING ENGINES AND REFRIGERATORS

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1. INTRODUCTION

The gas in an ideal Stirling engine does not execute either the two constant volume processes or the two constant temperature processes with heat transfer that constitute the Stirling cycle. A model for the operation of the ideal Stirling engine was devised by Schmidt [3]. In Schmidt’s model, the volumes of the compression and expansion spaces are taken to vary sinusoidally in time. The heat transfer processes in these spaces are assumed to occur at constant temperature. The pressure in the engine is taken to be spatially uniform. Schmidt’s theory provides simple analytical results, and has become the classic model for ideal Stirling engine operation. Because it does not take account of any dissipation, the predicted thermal efficiency equals the Carnot efficiency $\eta_{\text{Carnot}} = 1 - T_c/T_h$. The present paper considers the basic limitations on performance of Stirling engines and refrigerators that result from the pressure differences across the regenerator. Without these pressure differences there would be no flow through the regenerator, and the power output would be zero. The pressure oscillations in the compression and expansion spaces are taken to be sinusoidal in time. The analysis is carried out using linearized theory. The maximum attainable value of $P$ is determined by optimizing with respect to $\pi_c$ and $\delta$. It is shown that the power $P_{\text{reg}}$ dissipated in the regenerator is the product of the cold side temperature $T_c$ and the rate $(\delta)$ at which entropy is generated in the regenerator. By determining the entropy flow, it is found that $P_{\text{reg}}$ is removed as part of the heat $Q_c$ withdrawn at the cold side.

2. POWER OUTPUT AND THERMAL EFFICIENCY

The analysis is based on the one-dimensional model sketched in fig. 1. The regenerator is taken to be thermally perfect, and to have zero void volume. Friction of the pistons as well as heat losses to the walls are neglected. The flow in the high and low temperature spaces is assumed to be isentropic. Heat is supplied at the rate $Q_h$ at the high temperature side of the regenerator, and is withdrawn at the rate $Q_c$ at the low temperature side. Important results follow from the rate of energy balance for open systems

$$\frac{dU}{dt} = \dot{m}_h h_h - \dot{m}_c h_c + \dot{Q} - \dot{W}$$

(1)

Since the void volume of the regenerator is taken to be zero, $\dot{m}_c = \dot{m}_h = \dot{m}$. Application to the control volume just surrounding the exit plane $h$ of the regenerator, as well as to the control volume just enclosing the expansion space yields

$$\begin{pmatrix} W_e \\ \dot{W}_e \end{pmatrix} = \begin{pmatrix} \dot{Q}_h \\ \dot{m}_h \end{pmatrix} - c_p \left( \dot{m} T_o^* \right)$$

(2)

where $T_o^*$ is the time-varying part of the temperature just outside the exit plane $h$.

![Fig. 1. Sketch of model used for Stirling engine. The high temperature space is the expansion space, the low temperature one is the compression space](image-url)
The brackets denote average over a cycle. Use was made of the conditions \((m=0\) and \(T=T_0=\) constant constant just inside the exit plane \(h\). A similar application at the low temperature side yields

\[
(W_s) = (\dot{Q}_s) = c_v (mT_v^\prime)
\]

(3)

These expressions are evaluated by linearizing the pressures and mass flow rates

\[
 p_s = p_0 + p_s \sin(\omega t) \quad (p_s << p_0)
\]

(4)

\[
 p_s = p_0 + p_s \cos(\omega t + \delta) \quad (p_s << p_0)
\]

(5)

\[
 m = \rho_0 \dot{V}_s = \rho_0 \dot{V}_s = \rho_0 [\dot{p}_s(t) - \dot{p}_s(t)]
\]

(6)

The primes indicate first order quantities, while \(\rho_0\) is an appropriate average density. The unperturbed pressure \(p_0\) is constant throughout the two spaces and the regenerator. The linearized isentropic relation between temperature and pressure is:

\[
 T_s = \frac{\gamma-1}{\gamma} T_0 p_s \quad (T_s << T_0)
\]

(7)

As shown in the Appendix

\[
(W_s) = (\dot{Q}_s) = (p_s \dot{V}_s)
\]

(8)

\[
(W_s) = (\dot{Q}_s) = (p_s \dot{V}_s)
\]

(9)

Working out these expressions leads to

\[
(W_s) = (\dot{Q}_s) = \frac{1}{2} C \Delta \rho_0 \frac{T_s}{T_0} \pi c \cos(\delta) - 1
\]

(10)

\[
(W_s) = (\dot{Q}_s) = \frac{1}{2} C \Delta \rho_0 \frac{T_s}{T_0} \pi c \cos(\delta)
\]

(11)

Here, \(\pi_c = \frac{\Delta \rho_c}{\Delta \rho_0} (>1)\) is the ratio of pressure amplitudes in the two spaces. The corresponding power output of an ideal Stirling engine is:

\[
\dot{P} = (W_s) - (W_s) = \frac{1}{2} C \Delta \rho_0 \frac{T_s}{T_0} \pi c \cos(\delta)
\]

(12)

where the nondimensional power output \(P\) equals

\[
P(\pi, \delta) = \pi c \cos(\delta) \left[1 + \frac{T_s}{T_c} \right] - \left[1 + \frac{T_c}{T_s} \right] \pi c
\]

(13)

The corresponding value of the thermal efficiency is

\[
\eta = 1 - \frac{(W_s)}{(\dot{Q}_s)} = 1 - \frac{T_c}{T_s} \pi c \cos(\delta) - 1
\]

(14)

Since the coefficient of \(\cos(\delta)\) in equation (13) is positive, optimization of \(P\) with respect to \(\delta\) requires \(\cos(\delta) = 1\). Optimization with respect to \(\pi_c\) then yields

\[
\pi_{max} = \frac{1}{4} \left(\frac{T_c}{T_s} - \frac{T_s}{T_c}\right)^2
\]

(15)

for \(\pi_c = (1 + T_s/T_c)/2\). The corresponding value of the thermal efficiency is

\[
\eta(P = \pi_{max}) = (1 - T_c/T_s)/2 = \eta_{max}/2
\]

(16)

Figure 2 shows \(P\) and \(\eta/\eta_{\text{Carnot}}\) as function of the pressure amplitude ratio \(\pi_c\) at \(T_s/T_c = 4\) for the two cases \(\cos(\delta) = 1\) and \(\cos(\delta) = 0.9\). It is seen that both power output and thermal efficiency are quite sensitive to the value of \(\cos(\delta)\). For \(\cos(\delta) = 1\), the power output is negative for values of \(\pi_c\) near 1 and near \(T_s/T_c\).

\[\text{Fig. 2. Nondimensional power output } P \text{ and thermal efficiency } \eta/\eta_{\text{Carnot}} \text{ as function of ratio of pressure amplitudes } \pi_c. \text{ Solid curves are for } \cos(\delta)=1, \text{ dotted curves for } \cos(\delta)=0.9.\]

Note that the maximum value of the thermal efficiency \(\eta_{\text{Carnot}} = 1 - T_c/T_s\) is reached only for the case \(\delta = 0\) and \(\pi_c = 1\), in which case the power output \(P = 0\). Note also that the results shown in Fig. 2 are independent of the regenerator conductance \(C_r\).

The underlying reason is that the work and heat flow rates given by Eqs. (8) and (9) all are proportional to volume flow rate, and, hence, to \(C_r\) (eq.6). It must be concluded that Schmidt’s result for \(\eta\) at infinitely large \(C_r\) ([3, 9]) \(\eta = \eta_{\text{Carnot}} = 1 - T_c/T_s\) for arbitrary \(\pi_c\) and \(\delta\) represents a singular case.

### 3. POWER FLOW IN THE REGENERATOR

The power output of a Stirling engine without energy dissipation would be \(\dot{P}_{\text{Carnot}} = \eta_{\text{Carnot}} (\dot{Q}_s)\). Using Eq. (10), this can be expressed as

\[
\dot{P}_{\text{Carnot}} = \frac{1}{2} C \Delta \rho_0 \frac{T_s}{T_0} \pi c \cos(\delta)
\]

(17)

\[
P_{\text{Carnot}} = \left(1 - \frac{T_c}{T_s}\right) \left[\pi c \cos(\delta) - 1\right]
\]

(18)

The difference between Eq. (17) and the actual power output \(\dot{P}\) given by Eq. (12) is the power \(\dot{P}_{\text{reg}}\) dissipated in the regenerator.
Entropy enters the regenerator at the rate \(\tilde{\dot{Q}}_h/T_h\) on the hot side, and increases in value due to dissipation at the rate \(\tilde{\dot{\sigma}}\) while flowing to the cold side. It leaves the cold side at the rate \((\dot{Q}_c)/T_c\) which is the sum of the rate entering and the aggregate rate of dissipation. The value of \((\dot{\sigma})\) can be determined by applying Eq. (21) to an elementary slice \(dx\) of the regenerator (see fig.1). Upon averaging over a cycle this yields, to first order,

\[
d(\sigma) = (mdx) = \left[ m \left( c_p \frac{dT}{T} - R \frac{dp}{p} \right) \right] = -\left( \dot{V} \frac{dp}{T_c} \right)
\]

Evaluating the right hand side of this equation by using Eqs. (4)-(6) results in:

\[
\dot{\sigma} = \frac{1}{2} C_i A p_h^2 \frac{1}{T_w} \left( \pi_c^2 - 2 \pi_c \cos(\delta) + 1 \right)
\]

Comparison with Eqs. (19) and (20) shows that \(\tilde{\dot{\sigma}}\) in the regenerator.

4. ANALYSIS OF THE STIRLING REFRIGERATOR

The analysis of the Stirling refrigerator is nearly identical to that of the Stirling engine. The major difference is that the directions as well as the relative magnitudes of the heat flows are reversed (see fig.4).
As a consequence, the sign in Eq. (6) must be reversed: $m = \rho_v C_v (p_v^* - p^*_c)$. Another difference is that in this case $\pi_c = \Delta p_c / \Delta p_h < 1$. The rate of cooling at $T_c$ is given by
\[
\left( \dot{Q}_c \right) = \left( \dot{W}_c \right) = \frac{1}{2} C_v \Delta p_h^2 \frac{T_c}{T_c^*} (\dot{H}_c)
\]  
(27)
\[
\left( \dot{H}_c \right) = \frac{T_c}{T_c^*} \pi_c \cos(\delta - \pi_c)
\]  
(28)

The rate of heat withdrawal at $T_h$ by
\[
\left( \dot{Q}_h \right) = \left( \dot{W}_h \right) = \frac{1}{2} C_v \Delta p_h^2 \frac{T_h}{T_h^*} (\dot{H}_h)
\]  
(29)
\[
\left( \dot{H}_h \right) = 1 - \pi_c \cos(\delta)
\]  
(30)

It follows that
\[
\text{COP} = \frac{\left( \dot{Q}_c \right)}{\left( \dot{Q}_h \right) - \left( \dot{Q}_h \right)} = \frac{\frac{T_c}{T_c^*}}{1 + \pi_c^2 \frac{T_c}{T_c^*} - (1 + \frac{T_c}{T_c^*}) \pi_c \cos(\delta)}
\]  
(31)

Optimization of $\left( \dot{H}_c \right)$ requires $\cos(\delta) = 1$ and $\pi_c = 1/2$, yielding $\left( \dot{H}_c \right)_{\text{max}} = (T_c / T_h) / 4$ and
\[
\text{COP} \left( \dot{H}_c \right) = \left( \dot{H}_c \right)_{\text{max}} = \frac{1}{2 T_h / T_c - 1}
\]  
(32)

Figure 5 shows $\left( \dot{H}_c \right) T_c / T_h$ and COP/COP$_{\text{Carnot}}$ as function of ratio of pressure amplitudes $\pi_c$. Solid curves are for $\cos(\delta)=1$, dotted curves for $\cos(\delta)=0.9$.

The corresponding power input to a Stirling refrigerator without energy dissipation would be
\[
\left( \dot{Q}_h \right) = \left( 1 - \frac{T_h}{T_c} \right) \pi_c \left[ \cos(\delta) - \pi_c \right]
\]  
(34)

The difference between these two quantities is the nondimensional power dissipated in the regenerator
\[
P_{\text{reg}} = 1 - 2 \pi_c \cos(\delta) + \pi_c^2
\]  
(35)

The nondimensional power inputs are shown in Fig. 6 as function of the pressure amplitude ratio $\pi_c$ at $T_h / T_c = 4$ for $\cos(\delta) = 1$ and 0.9.

![Nondimensional cooling rate as function of ratio of pressure amplitudes](image)

5 CONCLUSION

The flow in the regenerator of a Stirling engine is driven by differences of pressure in the compression and expansion spaces. Taking account of the resulting energy dissipation leads to severe limitations on maximum attainable thermal efficiency and nondimensional power output $P$. These limitations are independent of the regenerator conductance $C_r$. It is concluded that Schmidt’s result $\eta = \eta_{\text{Carnot}}$ [3] obtained with $C_r = \infty$ represents a singular case. The thermal efficiency is independent of $C_r$ because the relevant work and heat flow rates all are proportional to volume flow rate (eqs. 8 and 9), and hence to $C_r$ (eq. 6). Optimum results are obtained at phase angle $\delta = 0$. At this phase angle, $\eta = \eta_{\text{Carnot}}$ is a linear function of the pressure amplitude ratio $\pi_c$. The value of $\eta$ goes from 1 at $\pi_c = 0$ to 0 at $\pi_c = T_h / T_c$ (fig. 2). The corresponding dependence of $P$ on $\pi_c$ is parabolic, with $P = 0$ at $\pi_c = 0$ and at $\pi_c = T_h / T_c$. The maximum value of $P$ is
\[ P_{max} = \left( \frac{T_h}{T_i} - \frac{T_i}{T_h} \right)^2 / 4. \] At \( P = P_{max} \), the value of the thermal efficiency is \((1 - T_i/T_h)/2\) (half the Carnot value). Inasmuch as engines are designed primarily for maximum power output, this constitutes a severe practical limitation. As a consequence, it is not surprising that actual Stirling engines have thermal efficiencies typically not exceeding 40\%. The power \( \dot{P}_{reg} \) dissipated in the regenerator equals the product of the cold side temperature \( T_c \) and the rate \((\dot{s})\) at which entropy is generated in the regenerator.

\( \dot{P}_{reg} \) is removed as part of the heat \((\dot{Q}_c)\) withdrawn at the cold side (eq. 26). The nondimensional power \( P_{reg} \) increases with \( \pi_c \) (fig. 3). The nondimensional rate of refrigeration \((\dot{H}_r)\) of a Stirling refrigerator has a parabolic dependence on \( \pi_c \) (eq. 27 and fig. 5). The maximum value of \( \dot{H}_r \) is \((T_c/T_h)^4\), achieved at \( \pi_c = 1/2 \). The corresponding value of the coefficient of performance COP is \(1/(1 - 2T_c/T_h)\), which is less than half of the Carnot value \(1/(1 + T_c/T_h)\). In the case of the Stirling refrigerator, \( \dot{P}_{reg} \) is removed as part of the heat \((\dot{Q}_c)\) withdrawn at the warm side (eq. 36), and \( \dot{P}_{reg} \) decreases as \( \pi_c \) increases (Fig. 6). The coefficient of performance is independent of \( C_r \) because the relevant heat and work flow rates all are proportional to volume flow rate, and hence to \( C_r \) (eqs. 27 and 29). The results obtained represent dissipation losses that are unavoidable when a regenerator is used to obtain mechanical power or refrigeration. Losses due to all other non-idealities have been neglected.

**REFERENCES**


(continuare de la pag. 67)

Astfel, s-au organizat:

CNT - 1 la Institutul de Construcții București (24 și 25 mai 1991);
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CNT - 8 la Universitatea din Pitești (29 și 30 mai 1998).

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