MAXIMUM ATTAINABLE PERFORMANCE OF STIRLING ENGINES AND REFRIGERATORS

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Rezumat. *Curgerea prin regeneratorul unei maşini Stirling depinde de diferenţele de presiune dintre spaţiul de compresie şi cel de destindere. Aceste diferenţe conduc la disipare de putere în regenerator. Folosind teoria linearizată, se arată că disiparea puterii limitează radamentul termic maxim care poate fi realizat precum şi puterea adimensională produsă. Valorile maxime posible nu depind de valoarea conductanţei regeneratorului. În cazul puterii maxime produse, randamentul termic este numai jumătate din valoarea randamentului Carnot. Puterea dispata in regenerator este eliminată ca parte a căldurii extrase la partea rece a regeneratorului. Sunt prezentate rezultate similare şi pentru maşinile frigorifice Stirling. La o valoare optimă a ratei de refrigerare adimensionalizate, coeficientul de performanţă are o valoare mai mică decât jumătate din cea a ciclului Carnot.*

1. INTRODUCTION

The gas in an ideal Stirling engine does not execute either the two constant volume processes or the two constant temperature processes with heat transfer that constitute the Stirling cycle. A model for the operation of the ideal Stirling engine was devised by Schmidt [3]. In Schmidt's model, the volumes of the compression and expansion spaces are taken to vary sinusoidally in time. The heat transfer processes in these spaces are assumed to occur at constant temperature. The pressure in the engine is taken to be spatially uniform. Schmidt's theory provides simple analytical results, and has become the classic model for ideal Stirling engine operation. Because it does not take account of any dissipation, the predicted thermal efficiency equals the Carnot efficiency $\eta_{Carnot} = 1 - T_c / T_h$. The present paper considers the basic limitations on performance of Stirling engines and refrigerators that result from the pressure differences across the regenerator. Without these pressure differences there would be no flow through the regenerator, and the power output would be zero. The pressure oscillations in the compression and expansion spaces are taken to be sinusoidal in time. The analysis is carried out using linearized theory. The maximum attainable value of P is determined by optimizing with respect to π_c and δ. It is shown that the power P_{reg} dissipated in the regenerator is the product of the cold side temperature T_c and the rate $(\dot{\sigma})$ at which entropy is generated in the regenerator. By determining the entropy flow, it is found that P_{reg} is removed as part of the heat $\dot{Q}_c$ withdrawn at the cold side.

2. POWER OUTPUT AND THERMAL EFFICIENCY

The analysis is based on the one-dimensional model sketched in fig. 1. The regenerator is taken to be thermally perfect, and to have zero void volume.

Friction of the pistons as well as heat losses to the walls are neglected. The flow in the high and low temperature spaces is assumed to be isentropic. Heat is supplied at the rate $\dot{Q}_h$ at the high temperature side of the regenerator, and is withdrawn at the rate $\dot{Q}_c$ at the low temperature side. Important results follow from the rate of energy balance for open systems

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \dot{m}_c h_c - \dot{m}_h h_h + \dot{Q} - \dot{W} \tag{1}$$

Since the void volume of the regenerator is taken to be zero, $\dot{m}_c = \dot{m}_h = \dot{m}$. Application to the control volume just surrounding the exit plane h of the regenerator, as well as to the control volume just enclosing the expansion space yields

$$\left(\dot{W}_h\right) = \left(\dot{Q}_h\right) = \left(\dot{m} h_h\right) = c_p \left(\dot{m} T_h^n\right) \tag{2}$$

where T_h^n is the time-varying part of the temperature just outside the exit plane h.

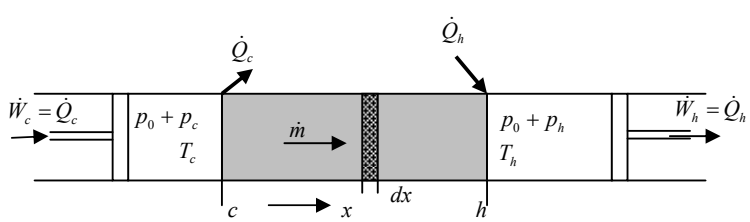

Fig. 1. Sketch of model used for Stirling engine. The high temperature space is the expansion space, the low temperature one is the compression space

The brackets denote average over a cycle. Use was made of the conditions $(\dot{m})=0$ and $T=T_h=\text{constant}$ constant just inside the exit plane *h*. A similar application at the low temperature side yields

$$\left(\dot{W}_c\right)=\left(\dot{Q}_c\right)=c_p\left(\dot{m}T_c^n\right) \tag{3}$$

These expressions are evaluated by linearizing the pressures and mass flow rates

$$p_h=p_0+p_h^{'}=p_0+\Delta p_h\sin(\omega t)\quad\left(p_h^{'}<<p_0\right) \tag{4}$$

$$p_c=p_0+p_c^{'}=p_0+\Delta p_c\sin(\omega t+\delta)\quad\left(p_c^{'}<<p_0\right) \tag{5}$$

$$\dot{m}=\rho_h\dot{V}_h=\rho_c\dot{V}_c=\rho_{av}C_r\left[p_c^{'}(t)-p_h^{'}(t)\right] \tag{6}$$

The primes indicate first order quantities, while ρ_{av} is an appropriate average density. The unperturbed pressure p_0 is constant throughout the two spaces and the regenerator. The linearized isentropic relation between temperature and pressure is:

$$T_h^{'}=\frac{\gamma-1}{\gamma}\frac{T_h}{p_0}p_h^{'}\quad\left(T_h^{'}<<T_h\right) \tag{7}$$

As shown in the Appendix

$$\left(\dot{W}_h\right)=\left(\dot{Q}_h\right)=\left(p_h^{'}\dot{V}_h\right) \tag{8}$$

$$\left(\dot{W}_h\right)=\left(\dot{Q}_h\right)=\left(p_h^{'}\dot{V}_h\right) \tag{9}$$

Working out these expressions leads to

$$\left(\dot{W}_h\right)=\left(\dot{Q}_h\right)=\frac{1}{2}C_r\Delta p_h^2\frac{T_h}{T_{av}}\left[\pi_c\cos(\delta)-1\right] \tag{10}$$

$$\left(\dot{W}_h\right)=\left(\dot{Q}_h\right)=\frac{1}{2}C_r\Delta p_h^2\frac{T_c}{T_{av}}\pi_c\left[\pi_c-\cos(\delta)\right] \tag{11}$$

Here, $\pi_c\equiv\frac{\Delta p_c}{\Delta p_h}(>1)$ is the ratio of pressure amplitudes in the two spaces. The corresponding power output of an ideal Stirling engine is:

$$\tilde{P}=\left(\dot{W}_h\right)-\left(\dot{W}_c\right)=\frac{1}{2}C_r\Delta p_h^2\frac{T_h}{T_{av}}P(\pi_c,\delta) \tag{12}$$

where the nondimensional power output *P* equals

$$P(\pi_c,\delta)=\pi_c\cos(\delta)\left(1+\frac{T_c}{T_h}\right)-\left(1+\frac{T_c}{T_h}\pi_c^2\right) \tag{13}$$

The corresponding value of the thermal efficiency is

$$\eta\equiv1-\frac{\left(\dot{Q}_c\right)}{\left(\dot{Q}_h\right)}=1-\frac{T_c}{T_h}\pi_c\frac{\pi_c-\cos(\delta)}{\pi_c\cos(\delta)-1} \tag{14}$$

Since the coefficient of $\cos(\delta)$ in equqtion (13) is positive, optimization of *P* with respect to δ requires $\cos(\delta)=1$. Optimization with respect to π_c then yields

$$P_{\max}=\frac{1}{4}\left(\sqrt{\frac{T_h}{T_c}}-\sqrt{\frac{T_c}{T_h}}\right)^2 \tag{15}$$

for $\pi_c=\left(1+T_h/T_c\right)/2$. The corresponding value of the thermal efficiency is

$$\eta\left(P=P_{\max}\right)=\left(1-T_c/T_h\right)/2=\eta_{Carnot}/2 \tag{16}$$

Figure 2 shows *P* and η/η_{Carnot} as function of the pressure amplitude ratio π_c at $T_h/T_c=4$ for the two cases $\cos(\delta)=1$ and $\cos(\delta)=0.9$. It is seen that both power output and thermal efficiency are quite sensitive to the value of $\cos(\delta)$. For $\cos(\delta)=1$, the power output is negative for values of π_c near 1 and near T_h/T_c.

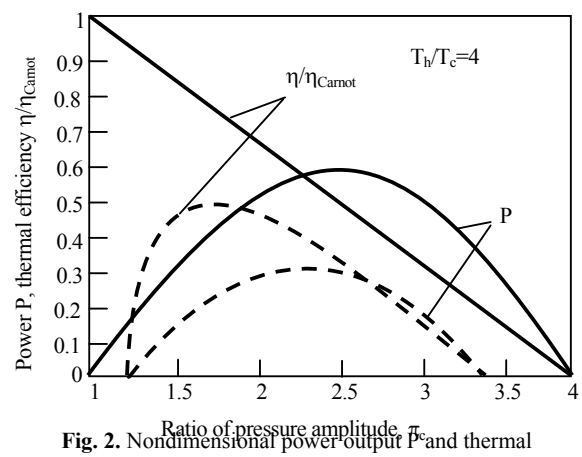

Fig. 2. Nondimensional power output P and thermal efficiency η/η_{Carnot} as function of ratio of pressure amplitudes π_c. Solid curves are for cos (δ)=1, dotted curves for cos (δ)=0.9

Note that the maximum value of the thermal efficiency $\eta_{Carnot}=1-T_c/T_h$ is reached only for the case $\delta=0$ and $\pi_c=1$, in which case the power output $P=0$. Note also that the results shown in Fig. 2 are independent of the regenerator conductance C_r.

The underlying reason is that the work and heat flow rates given by Eqs. (8) and (9) all are proportional to volume flow rate, and, hence, to C_r (eq.6). It must be concluded that Schmidt's result for η at infinitely large C_r ([3, 9] $\eta=\eta_{Carnot}=1-T_c/T_h$ for arbitrary π_c and δ) represents a singular case.

3. POWER FLOW IN THE REGENERATOR

The power output of a Stirling engine without energy dissipation would be $\tilde{P}_{Carnot}=\eta_{Carnot}\left(\dot{Q}_h\right)$. Using Eq. (10), this is can be expressed as

$$\tilde{P}_{Carnot}=\frac{1}{2}C_r\Delta p_h^2\frac{T_h}{T_{av}}P_{Carnot} \tag{17}$$

$$P_{Carnot}=\left(1-\frac{T_c}{T_h}\right)\left[\pi_c\cos(\delta)-1\right] \tag{18}$$

The difference between Eq. (17) and the actual power output $\tilde{P}$ given by Eq. (12) is the power $\tilde{P}_{reg}$ dissipated in the regenerator

$$\tilde{P}_{reg} = \frac{1}{2} C_r \Delta p_h^2 \frac{T_h}{T_{av}} P_{reg} \tag{19}$$

$$P_{reg} = \frac{T_c}{T_h}\left[\pi_c^2 - 2\pi_c \cos(\delta) + 1\right] \tag{20}$$

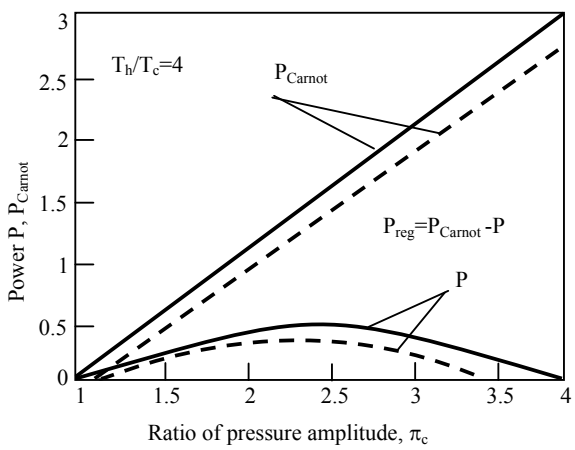

Fig. 3. Nondimensional power outputs P_{Carnot} and P as function of ratio of pressure amplitudes π_c . Solid curves are for cos (δ)=1, dotted curves for cos (δ)=0.9

The nondimensional powers P_{Carnot} and P are shown in Fig. 3. The difference between the two is the nondimensional power P_{reg} dissipated in the regenerator. The power $\tilde{P}_{reg}$ is removed as part of the heat flow $(\dot{Q}_c)$.

This can be shown by applying the rate of entropy balance equation

$$\frac{dS}{dt} = \int \frac{\delta \dot{Q}}{T} + \dot{m}(s_c - s_h) + \dot{\sigma} \tag{21}$$

to the control volume just surrounding the regenerator. Averaging over a cycle yields

$$\frac{(\dot{Q}_c)}{T_c} = \frac{(\dot{Q}_h)}{T_h} + (\dot{\sigma}) \tag{22}$$

The term $\dot{m}(s_c - s_h)$ does not contribute to this result because the flows in the compression and expansion spaces are isentropic, and because $\dot{m} = 0$.

Entropy enters the regenerator at the rate $(\dot{Q}_h)/T_h$ on the hot side, and increases in value due to dissipation at the rate $(\dot{\sigma})$ while flowing to the cold side. It leaves the cold side at the rate $(\dot{Q}_c)/T_c$ which is the sum of the rate entering and the aggregate rate of dissipation. The value of $(\dot{\sigma})$ can be determined by applying Eq. (21) to an elementary slice dx of the regenerator (see fig.1). Upon averaging over a cycle this yields, to first order,

$$d(\dot{\sigma}) = (\dot{m} ds) = \left[\dot{m}\left(c_p \frac{dT}{T} - R\frac{dp}{p}\right)\right] = -\frac{(\dot{V}_c dp)}{T_c} \tag{23}$$

Use was made of the circumstance that the temperature difference dT between the two sides of the slice is constant. Use was also made of the relations $\dot{m} = \rho_c \dot{V}_c$ and $p_0 = \rho_c R T_c$. Since $\dot{V}_c$ is independent of x, integration of Eq. (23) over the length of the regenerator yields

$$(\dot{\sigma}) = -\frac{1}{T_c}\int_c^h (\dot{V}_c dp) = \frac{1}{T_c}\left[\dot{V}_c\left(p_c' - p_h'\right)\right] \tag{24}$$

Evaluating the right hand side of this equation by using Eqs. (4)-(6) results in:

$$(\dot{\sigma}) = \frac{1}{2} C_r \Delta p_h^2 \frac{1}{T_{av}}\left(\pi_c^2 - 2\pi_c \cos(\delta) + 1\right) \tag{25}$$

Comparison with Eqs. (19) and (20) shows that $\tilde{P}_{reg} = T_c(\dot{\sigma})$, and hence that:

$$(\dot{Q}_c) = (\dot{Q}_h)\frac{T_c}{T_h} + \tilde{P}_{reg} \tag{26}$$

It follows that $\tilde{P}_{reg}$ arises from the rate at which entropy is generated by irreversibilities in the regenerator.

4. ANALYSIS OF THE STIRLING REFRIGERATOR

The analysis of the Stirling refrigerator is nearly identical to that of the Stirling engine. The major difference is that the directions as well as the relative magnitudes of the heat flows are reversed (see fig.4).

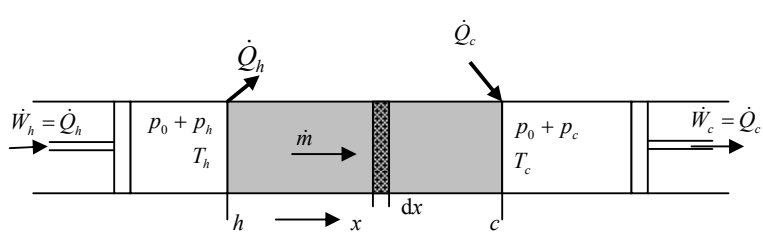

Fig. 4. Sketch of model used for Stirling refrigerator

As a consequence, the sign in Eq. (6) must be reversed: $\dot{m} = \rho_{av} C_r \left(p_h^{'} - p_c^{'} \right)$. Another difference is that in this case $\pi_c \equiv \Delta p_c / \Delta p_h < 1$. The rate of cooling at T_c is given by

$$\left(\dot{Q}_c\right) = \left(\dot{W}_c\right) = \frac{1}{2} C_r \Delta p_h^2 \frac{T_h}{T_{av}} \left(\dot{H}_c\right) \tag{27}$$

$$\left(\dot{H}_c\right) = \frac{T_c}{T_h} \pi_c \left[\cos(\delta) - \pi_c\right] \tag{28}$$

the rate of heat withdrawal at T_h by

$$\left(\dot{Q}_h\right) = \left(\dot{W}_h\right) = \frac{1}{2} C_r \Delta p_h^2 \frac{T_h}{T_{av}} \left(\dot{H}_h\right) \tag{29}$$

$$\left(\dot{H}_h\right) = 1 - \pi_c \cos(\delta) \tag{30}$$

It follows that

$$COP \equiv \frac{\left(\dot{Q}_c\right)}{\left(\dot{Q}_h\right) - \left(\dot{Q}_c\right)} = \\ = \frac{T_c}{T_h} \frac{\pi_c \left[\cos(\delta) - \pi_c\right]}{1 + \pi_c^2 T_c / T_h - \left(1 + T_c / T_h\right) \pi_c \cos(\delta)} \tag{31}$$

Optimization of $\left(\dot{H}_c\right)$ requires $\cos(\delta) = 1$ and $\pi_c = 1/2$, yielding $\left(\dot{H}_c\right)_{\max} = \left(T_c / T_h\right)/4$ and

$$COP\left(\left(\dot{H}_c\right) = \left(\dot{H}_c\right)_{\max}\right) = \frac{1}{2 T_h / T_c - 1} \tag{32}$$

Figure 5 shows $\left(\dot{H}_c\right) T_h / T_c$ and $COP / COP_{Carnot} \equiv$ $\equiv COP\left(T_h / T_c - 1\right)$ as function of π_c at $T_h / T_c = 4$ for the two cases $\cos(\delta) = 1$ and $\cos(\delta) = 0.9$.

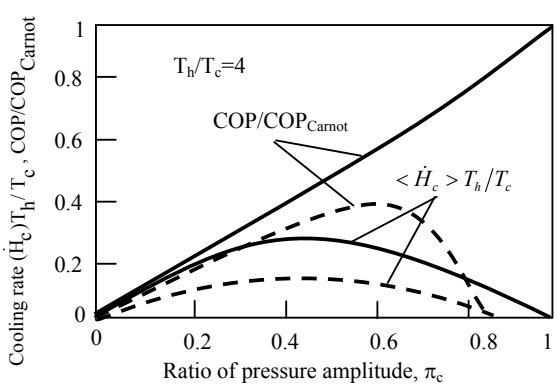

Fig. 5. Nondimensional cooling rate $\left(\dot{H}_c\right) T_h / T_c$ and coefficient of performance COP/COP$_{Carnot}$ as functions of ratio of pressure amplitudes π_c. Solid curves are for cos (δ)=1, dotted curves for cos (δ)=0.9.

The nondimensional power input to the Stirling refrigerator is:

$$\left(\dot{H}_h\right) - \left(\dot{H}_c\right) = 1 + \frac{T_c}{T_h} \pi_c^2 - \left(1 + \frac{T_c}{T_h}\right) \pi_c \cos(\delta) \tag{33}$$

The corresponding power input to a Stirling refrigerator without energy dissipation would be

$$\left(\frac{T_h}{T_c} - 1\right)\left(\dot{H}_c\right) = \left(1 - \frac{T_c}{T_h}\right) \pi_c \left[\cos(\delta) - \pi_c\right] \tag{34}$$

The difference between these two quantities is the nondimensional power dissipated in the regenerator

$$P_{reg} = 1 - 2\pi_c \cos(\delta) + \pi_c^2 \tag{35}$$

The nondimensional power inputs are shown in Fig. 6 as function of the pressure amplitude ratio π_c at $T_h / T_c = 4$ for $\cos(\delta) = 1$ and 0.9.

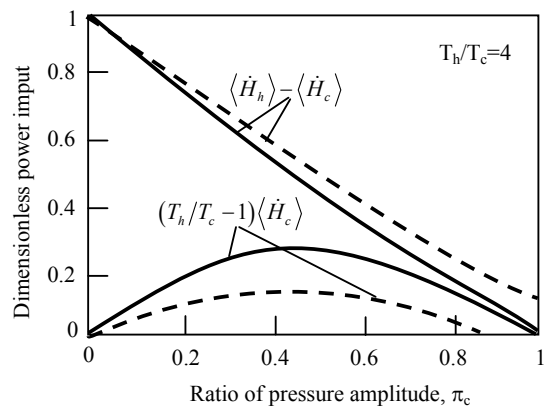

Fig. 6. Nondimensional cooling rate $\langle \dot{H}_h \rangle - \langle \dot{H}_c \rangle$ and $\left(T_h / T_c - 1\right)\langle \dot{H}_c \rangle$ as functions of ratio of pressure amplitudes π_c. Solid curves are for cos (δ)=1, dotted curves for cos (δ)=0.9.

In the case of the Stirling refrigerator, the entropy flow is from the cold to the hot side (see also [13]), and

$$\left(\dot{Q}_h\right) = \left(\dot{Q}_c\right) \frac{T_h}{T_c} + \tilde{P}_{reg} \tag{36}$$

5 CONCLUSION

The flow in the regenerator of a Stirling engine is driven by differences of pressure in the compression and expansion spaces. Taking account of the resulting energy dissipation leads to severe limitations on maximum attainable thermal efficiency hand nondimensional power output *P*. These limitations are independent of the regenerator conductance C_r. It is concluded that Schmidt's result $\eta = \eta_{Carnot}$ [3] obtained with $C_r = \infty$ represents a singular case. The thermal efficiency is independent of C_r because the relevant work and heat flow rates all are proportional to volume flow rate (eq. 8 and 9), and hence to C_r (eq. 6). Optimum results are obtained at phase angle $\delta = 0$. At this phase angle, η / η_{Carnot} is a linear function of the pressure amplitude ratio π_c. The value of η / η_{Carnot} goes from 1 at $\pi_c = 0$ to 0 at $\pi_c = T_h / T_c$ (fig. 2). The corresponding dependence of *P* on π_c is parabolic, with $P = 0$ at $\pi_c = 0$ and at $\pi_c = T_h / T_c$. The maximum value of *P* is

$P_{\max} = \left(\sqrt{T_h/T_c} - \sqrt{T_c/T_h}\right)^2 \Big/ 4$. At $P = P_{\max}$, the value of the thermal efficiency is $\left(1 - T_c/T_h\right)/2$ (half the Carnot value). Inasmuch as engines are designed primarily for maximum power output, this constitutes a severe practical limitation. As a consequence,

It is not surprising that actual Stirling engines have thermal efficiencies typically not exceeding 40%. The power $\tilde{P}_{reg}$ dissipated in the regenerator equals the product of the cold side temperature T_c and the rate $(\dot{\sigma})$ at which entropy is generated in the regenerator. $\tilde{P}_{reg}$ is removed as part of the heat $\left(\dot{Q}_c\right)$ withdrawn at the cold side (eq. 26). The nondimensional power P_{reg} increases with π_c (fig. 3). The nondimensional rate of refrigeration $\left(\dot{H}_c\right)$ of a Stirling refrigerator has a parabolic dependence on π_c (eq. 27 and fig. 5). The maximum value of $\left(\dot{H}_c\right)$ is $\left(T_c/T_h\right)/4$, achieved at $\pi_c = 1/2$. The corresponding value of the coefficient of performance COP is $1/\left(-1 + 2T_h/T_c\right)$, wich is less than halh of the Carnot value $1/\left(-1 + T_h/T_c\right)$. In the case of the Stirling refrigerator, $\tilde{P}_{reg}$ is removed as part of the heat $\left(\dot{Q}_h\right)$ withdrawn at the warm side (eq. 36), and P_{reg} decreases as π_c increases (Fig. 6). The coefficient of performance is independent of C_r because the relevant heat and work flow rates all are proportional to volume flow rate, and hence to C_r (eqs. 27 and 29). The results obtained represent dissipation losses that are unavoidable when a regenerator is used to obtain mechanical power or refrigeration. Losses due to all other non-idealities have been neglected.

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CNT - 1 la Institutul de Construcţii Bucureşti (24 şi 25 mai 1991);

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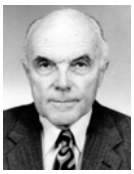

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