A NUMERICAL SOLUTION FOR A PLANE PROBLEM OF ELASTICITY

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Rezumat. Problemele de elasticitate plană care îndeplinesc cerințele rezolvării cu ajutorul ecuației biarmonice a elasticității plane pot fi soluționate, în numeroase cazuri, utilizând integrarea ecuației cu metoda diferențelor finite. În lucrare se prezintă o extindere a domeniului de aplicare a acestei ecuații la domenii dublu conexe (diafragmă octogonală cu gol central pătrat). Rezultatele obținute (rel. 13) sunt comparate cu cele date de utilizarea metodei elementelor finite (rel. 14) şi se constată o foarte bună corespondență între valorile tensiunilor obținute prin cele două metode. În concluzie, generalizând interpretarea mecanică a funcției de tensiuni şi a derivatei sale normale, se extinde domeniul de utilizare a ecuației elasticității plane la rezolvarea problemelor inginereşti. Precizia metodei diferențelor finite este mai bună ca cea a metodei elementelor finite deoarece modelul matematic folosit în primul caz este cel corespunzător funcțiilor continue, aproximația de calcul fiind de natură matematică şi aceasta poate fi îmbunătățită, până la limita dorită, în cadrul unui calcul mult mai redus. Lucrarea conține 5 figuri referitoare la diafragma analizată şi relațiile de calcul pentru tensiunile normale din două puncte caracteristice ale elementului.

Cuvinte cheie: metode numerice, funcția Airy, diferențe finite, tensiuni.

Abstract: Plane elasticity problems which fulfil the solving requirements with the help of the biharmonic equation of plane elasticity can be solved, in many cases, using the integration of the equation with finite differences method. In this paper is presented an extension of the field of application of this equation in double conex domains (octogonal diaphragm with a central square gap). The results obtained (rel. 13) are compaired with the ones given by the use of finite differences method (rel. 14) and it can be seen a very good correspondence between the values of the tensions obtained with the two methods. In conclusion, generalizing the mechanic interpretation of the tension function and of its normal derivative, the utilization domain of the plane elasticity equation in solving engineering problems is extended. The precision of the finite differences method is better than the one of finite elements method because the mathematic model used in the first case is the one appropriate for continuous functions, the approximation being of mathematic nature and this can be improved, until the desired limit, in a more reduced calculus. The paper contains 5 figures regarding the analyzed diaphragm and the calculus relations for the normal tensions from two characteristic points of the element.

Keywords: numerical methods, Airy’s function, finite differences, stresses, plane slab

1. INTRODUCTION

Numerical methods for the solving of a plane elasticity problem may be classified in two important groups:

– methods for numerical integration of elasticity differential equations, which are based on the domain discretization into elementary portion continuously connected one with the other and consequently, the calculation approximation is purely mathematical;

– method based on another physical model, the problem domain being divided into finite portions interconnected in certain points only, and consequently the calculation error is of a physical nature mainly (sometimes it may be accompanied by a mathematical one caused by solving of equation system).

Lately, methods from the second group have been especially developed. That is because the conditions of existence and uniqueness of the problem solution are less restrictive than these required by differential equations; the calculus volume (in most cases very laborious) being assumed by the electronic ordinators.

This paper brings to expert's attention the still usefulness of methods from the first group. That, at least, from two considerations:

– the calculations accuracy is bigger because the elementary physical model „describes" with fidelity the structure deformation phenomenon;

– the calculus volume becomes, in many cases, lower comparative with another procedure, which constitutes an important advantage of this method. To justify all these ideas we shall follow the stresses determining in a plane slab of constant thickness with the octagonal boundary and a central gap, charged on its long sides with a uniformly distributed loading (Fig. 1).

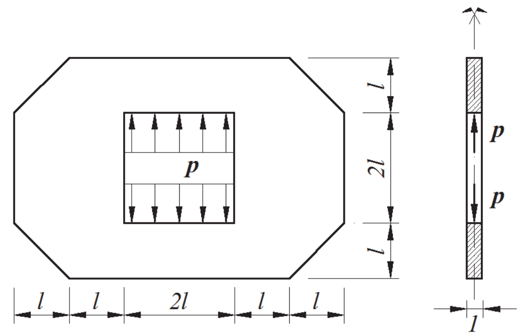


Fig. 1.

This slab represents the main strength element of a great capacity hydrodynamic press. The very impressive forces which charged this slab had elicted an ample theoretical and practical investigation program. For instance, the determining of stresses and displacements was made by the finite element method using a discretization network with two hundred points, which led to a great calculation volume. We would like to show that good results may be obtained in a more simply way; solving the problem by the plane elasticity equation. In this view, we shall present the determining of stresses in certain points (for instance, b and d) using this equation.

2. CALCULATION HYPOTHESES

a) The material of the slab is a continuous, homogenous and isotropic medium.

b) The displacements are small comparative to the slab dimensions and consequently, the equilibrium may be written on un-deformed position.

c) It is assumed that Hook's lineary-elastic law is obeyed.

3. THE MECHANIC INTERPRETATION GENERALIZATION OF AIRY'S FUNCTION

For applying the plane elasticity equation $\Delta\Delta F=0$, in which Airy's function F has the property

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \qquad \sigma_y = \frac{\partial^2 F}{\partial x^2} \qquad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \qquad (1), (2), (3)$$

it is necessary to know F and its normal derivative $\dfrac{\partial F}{\partial n}$ on the slab boundaries.

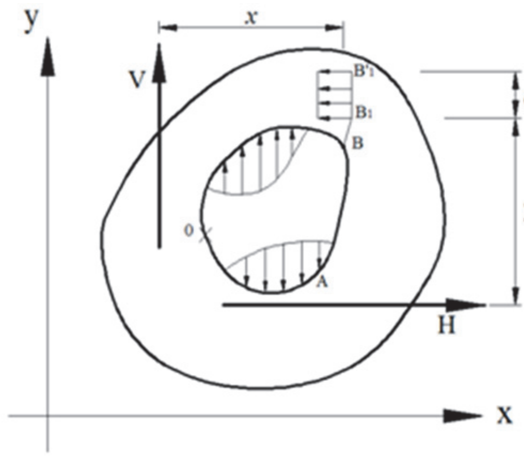


Fig. 2

Refer to the slab in the Figure 2 subjected on the interior boundary to a planar forces system p, situated in the slab middle plane. Consider O an arbitrary origin and BB_1 an incomplete section going to slab current point B_1. The forces resultant (external and internal) on the distance OAB_1 has the components V and H. The moment of these components in relation to B_1 is written:

$$M_{B_1} = H_y - V_x \tag{a}$$

Let's prolong the section form B_1 to B_1' situated at elementary distance dy from B_1. The moment in relation to B_1' forces applied on the length $OABB_1$' becomes:

$$M'_{B1} = H(y + \mathrm{d}y) - V_x + \sigma_x \frac{\mathrm{d}y^2}{2} \tag{b}$$

Neglecting the last term, from the relationships (a), (b) it results:

$$M'_{B_1} - M_{B_1} = H\mathrm{d}y$$

which leads to

$$H = \frac{\partial M}{\partial y} \tag{c}$$

At the same time the product $\sigma_x \mathrm{d}y$ represents the forces elementary variation H with respect to y:

$$\sigma_x dy = \frac{\partial H}{\partial y}\mathrm{d}y \tag{d}$$

From (c) and (d) relationships we deduct:

$$\sigma_x = \frac{\partial^2 M}{\partial y^2} \tag{4}$$

In the same manner, prolonging the section from B_1 to a point situated at elementary distance dx at B_1, we obtain:

$$\sigma_y = \frac{\partial^2 M}{\partial x^2} \tag{5}$$

The relationships (4), (5) having the identic form with (1), (2), it result that the stress function F in a point of plane slab, represents the moment with respect to that point of the forces (internal and external) applied on a distance which begins from an arbitrary origin and it is continued until the respective point F differs from Mby a first order polynom which may be neglected because it does not generate stresses.

If in boundary conditions

$$p_x = \sigma_x I_x + \tau_{yx} I_y \qquad p_y = \tau_{xy} I_x + \sigma_y I_y \tag{e)(f}$$

we introduce the relationships (1), (2), (3) and take into account that $I_x = \frac{dy}{\mathrm{d}s}$ and $I_y = -\frac{\mathrm{d}x}{\mathrm{d}s}$ it results:

$$p_x = \frac{\partial^2 F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial^2 F}{\partial x \partial y}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial y}\right) \qquad p_y = \frac{-\partial^2 F}{\partial x \partial y}\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\partial^2 F}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{\partial}{\partial s}\left(\frac{\partial F}{\partial x}\right)$$

Integrating these expressions on the length OAB we get

$$\frac{\partial F}{\partial x} = -R_y, \frac{\partial F}{\partial y} = R_x \tag{6) (7}$$

R_y and R_x being the forces resultant components on the axis y, respectively x, concerning this distance.

Orientating the reference system after the normal n and tangent t at the boundary, the relationships (6), (7) become:

$$\frac{\partial F}{\partial n} = N \qquad \frac{\partial F}{\partial t} = T \tag{8) (9}$$

which means that: the stress function derivative with respect to normal direction n in any point of distance OAB_1 is equal to the tangent component at the boundary from the same point of the forces applied from the origin O to the respective point. This component will be named „axial force".

Similarly, from the relationship (9) it results that the stress function derivative with respect to tangent direction t, in any point of the distance OAB_1, is equal to the normal component at the boundary from the same point, of the forces applied from origin O to respective point. This component will be named „shear force".

Thus, it is arrived at the generalization of mechanic interpretation of Airy's function and its derivatives for double-conex domains. This result makes possible its determination on any slab contour subjected to an equilibrium system forces.

Really, if the point B_1 from Figure 2, is situated on one of the slab contours and considering the origin O in any point of the same contour, in the determination of F and N, there join the external forces only and consequently it is arrived at the following practical calculation rule: considering each contour as the frame axis, open in any point and subjected to the external forces, we construct the moment and axial force diagrams; these represents, in fact, the stress function and its normal derivative values, lengthways of contour considered.

4. THE STRESSES DETERMINING IN PLANE SLAB

Using this conclusions, for the plane slab shown in Figure 1 we get:

– on the exterior contour, $F = 0$ and $\frac{\partial F}{\partial n} = 0$;

– on the interior contour, F and $\frac{\partial F}{\partial n}$ values, from Figure 3.

The equation $\Delta\Delta F = 0$ is integrated by the finite difference method. Choosing an equal step network λ and having in view the slab symmetry, this equation, written in points 1, 2 (Fig. 4) becomes:

$$\begin{aligned} &1 \Rightarrow 20F_1 - 8(F_2 + F_g + F_h + F_c) + 2(F_a + F_{1''} + F_f + F_b) + F_1 + F_{1'''} + F_{1'} + F_d = 0 \\ &2 \Rightarrow 20F_2 - 8(2F_1 + F_a + F_b) + 2(2F_h + 2F_c) + 2F_g + F_{2'} + F_{2''} = 0 \end{aligned} \tag{g}$$

But $F_a = F_h = F_g = F_f = F_e = F_d = 0, \quad F_c = F_b = \frac{pl^2}{2}$ (from Fig. 3).

At the same time $\frac{F_{1'} - F_1}{2\lambda} = N_h = 0 \Rightarrow F_1 = F_1$; that is: the exterior points of contour where $N = 0$, are symmetrical with the interior ones:

1' = 1 1" = 1 1"' = 1 2' = 2

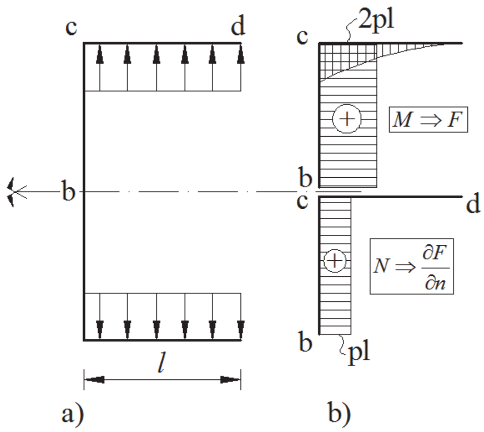


Fig. 3

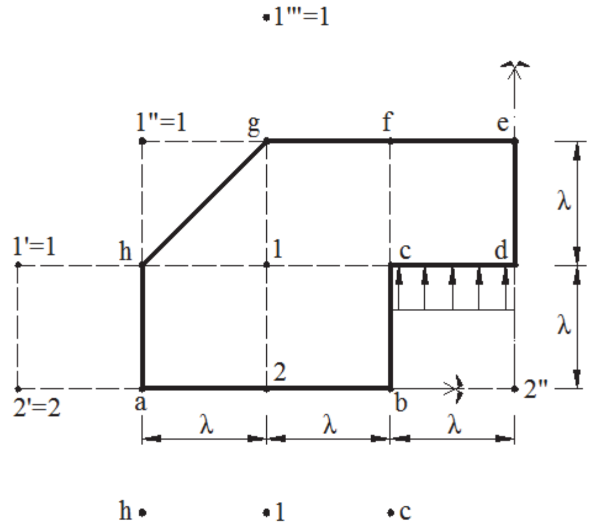


Fig. 4

In point b we have $\frac{F_{2''l} - F_2}{2\lambda} = pl \Rightarrow F_{2''l} = 2pl\lambda + F_2$

Similarly in point d we get $\frac{F_{2''h} - F_e}{2\lambda} = 0 \Rightarrow F_{2''h} = F_e = 0$

Taking for $F_{2''}$ an average value, it obtains $F_{2''} = \frac{F_{2''l} - F_{2''h}}{2} = pl^2 + \frac{F_2}{2}$

With this conditions the system (g) becomes: $\begin{cases} 25F_1 - 8F_2 = 3pl^2 \\ -16F_1 + 21{,}5F_2 = pl^2 \end{cases}$

Its solution is: $F_1 = 0{,}177\,pl^2$ $F_2 = 0{,}178\,pl^2$

Let's determine the stresses values in any two points of slab, for example, b and d:

- point b: $\sigma_{xb} = \frac{2(F_c - F_b)}{\lambda^2} = 2(0{,}5 - 0{,}5)p = 0$

$$\sigma_{yb} = \frac{F_2 - 2F_b + F_{2''e}}{\lambda^2} = (0{,}178 - 2 \cdot 0{,}5 + 2 + 0{,}178)p = 1{,}35\,p \tag{11}$$

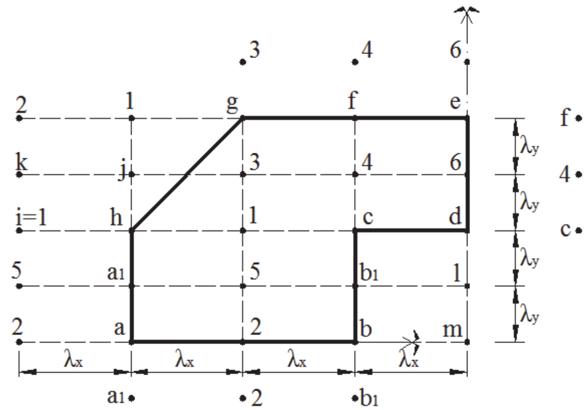
- point d: $\sigma_{xd} = \frac{F_e - 2F_d + F_{2''h}}{\lambda^2} = 0$

$$\sigma_{yd} = \frac{2(F_c - F_b)}{\lambda^2} = 2 \cdot 0{,}5\,p = p$$

We can increase the results precision repeating the calculation on the network with an unequal step (Fig. 5).

$$\lambda_x = \lambda \qquad \lambda_y = \lambda / 2$$

Fig. 5



In the same way we obtain:

$1 \Rightarrow 24{,}5F_1 + 2F_2 - 12F_3 + 2F_4 - 12F_5 = 2pl^2$

$2 \Rightarrow 4F_1 + 24F_4 - 24F_5 = 0$

$3 \Rightarrow -12{,}75F_1 + 0{,}25F_2 + 25F_3 - 6F_4 + 2F_5 + 0{,}5F_6 = -pl^2$

$4 \Rightarrow 2{,}25F_1 - 6F_3 + 25{,}5F_4 - 6F_6 = 5pl^2$

$5 \Rightarrow -12F_1 - 12F_2 + 2F_3 + 25{,}66F_5 + 0{,}33F_6 = 0{,}666pl^2$

$6 \Rightarrow F_3 - 12F_4 + 0{,}66F_5 + 26{,}33F_6 = -3{,}333pl^2$

The system solution is

$$\begin{array}{lll} F_1 = 0{,}177pl^2 & F_2 = 0{,}138pl^2 & F_3 = 0{,}081pl^2 \\ F_4 = 0{,}188pl^2 & F_5 = 0{,}167pl^2 & F_6 = -0{,}048pl^2 \end{array} \qquad (12)$$

The stresses values in the same points b and d result:

$$\begin{array}{ll} \bullet \text{ point b:} & \sigma_{xb} = 2(0{,}5 - 0{,}5)p = 0 \\ & \sigma_{yb} = (0{,}138 - 2 \cdot 0{,}5 + 0{,}138 + 2)p = 1{,}28\,p \\ \bullet \text{ point d:} & \sigma_{xd} = (-0{,}048 - 0{,}048)4\,p = -0{,}384\,p \\ & \sigma_{yd} = 2 \cdot 0{,}05\,p = p \end{array} \qquad (13)$$

Remark. For the considered plane slab, the stresses values in point d and b, determined by the finite element method, are:

$$\begin{array}{ll} \sigma_{xb} = 0 & \sigma_{xd} = -0{,}45\,p \\ \sigma_{yb} = 1{,}25\,p & \sigma_{yd} = p \end{array} \qquad (14)$$

The value obtained (11) and (13) applying the plane elasticity equation are very nearly at those which result using the finite element method (14) but the calculus is much more simple (in point d, the stress σ_{xd}, in the first calculation stage resulted zero because the network was very roughly).

5. CONCLUSIONS

Generalizing the mechanic interpretation of Airy's function and its normal derivative at double-conex domains the plane elasticity equation application field extends in solving many engineering problems.

Using the equation $\Delta\Delta F = 0$ the calculus volume remains in easy access limits and may be shown up by hand or with a pocket calculator.

The calculation precision is superior in comparison with other methods because the differential equations hold the privilege to render with fidelity the deformation phenomenon of strength structures.

Reference

[1] Precupanu, D.: *Teoria elasticității*. Editura I. P. I, Iași, 1982.
[2] Salvadori, M., G.: *Metode numerice în tehnică*. Editura Tehnică, București, 1972.
[3] Soare, M.: *Aplicarea ecuațiilor cu diferente finite la calculul plăcilor curbe subțiri*. Editura Academiei, București, 1968.
[4] Wang, P., C.: *Metode numerice și matriceale în mecanica construcțiilor*. Editura Tehnică, București, 1970.