ABOUT THE APPLICATION OF THE FINITE DIFFERENCE METHOD AT THE PROBLEM DEFINED ON THE DOUBLE CONEX DOMAINS

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Rezumat. Se prezintă analiza unei plăci plane încărcate cu o forță uniform distribuită, cu ajutorul metodei numerice a diferențelor finite. Problema, care se rezumă la determinarea tensiunilor în anumite puncte ale plăcii (stabilite la începulutul calculului), se rezolvă cu ajutorul ecuației elasticității plane. Tensiunile determinate în punctele propuse, cu ajutorul diferențelor finite, rezultă foarte apropiate ca valoare cu cele determinate prin metoda elementului finit.

Cuvinte cheie: metode numerice, funcția Airy, diferențe finite, tensiuni, placă plană.

Abstract. It is presented the analyses of a plane slab subjected to a uniform load, with the help of the finite differences numerical method. The problem, that summarizes at determining the stresses in certain points of the slab (assigned at the begining of the calculus), can be solved using the plane elasticity equation. The stresses determined in the established points, with the finite differences method, result very close, as value, with the ones determined by the fnite element method.

Keywords: numerical methods, Airy's function, finite differences, stresses, plane slab.

1. INTRODUCTION

Numerical methods for solving of plane elasticity problem may be classified in two important groups:

- methods for numerical integration of elasticity differencial equations, which are based on the domain discretization into elementary portion continuously connected one with the other and consequently, the calculation approximation is purely mathematical;
- methods based on another physical model, the problem domain being divided into finite portions interconnected in certain points only, and consequently the calculation error is of a physical nature mainly (sometimes it may be accompanied by a mathematical one caused by solving of equations system).

Lately methods from the second group have been especially developed. That is so because the conditions of existence and uniqueness of the problem's solution are less restrictive than these required by differencial equations; the calculus volume (is most cases very laborious) being assumed by the electronic ordinators.

This paper brings to the experts attention the still usefulness of the methods from the first group. That, at least, from two considerations:

- the calculation accuracy is bigger because the elementary physical model „describes" with fidelity the structure deformation phenomenon;

- the octagonal boundary and a central gap, charged on its long sides with uniformly distributed loading (fig. 1). This slab represents the main strength element of a great capacity hydrodynamic press. The very impressive forces which charged this slab had elicted an ample theoretic and practice investigation program.

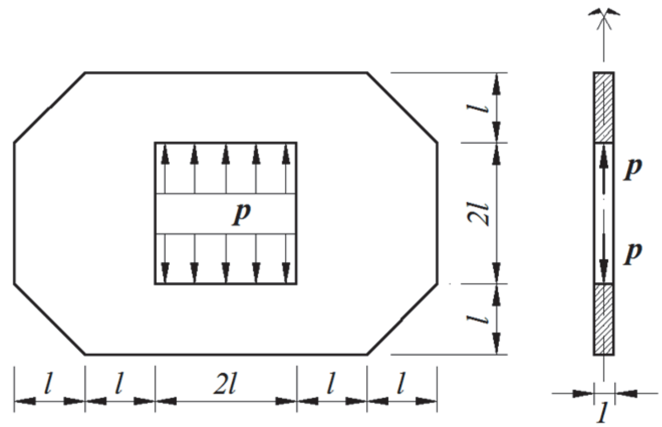


Fig. 1

For instance, the determining of stresses and displacements was made by finite element method using a discretization network with two hundred points, which led to a great calculation volume. We would like to show that good results may be obtained in a more simple way; solving the problem by the plane elasticity equation. In this view, we shall present the determining of stresses in certain points (for instance, b and d), using this equation.

2. CALCULATION HYPOTHESES

a) The material of the slab is continuous, homogenuous and isotropic medium;

b) The displacements are small comparative to the slab dimensions and consequently, the equilibrium may be written on undeformed position;

c) It is assumed that Hook's lineary-elastic law is obeyed.

3. THE MECHANIC INTERPRETATION GENERALIZATION OF AIRY'S FUNCTION

For applying the plane elasticity equation $\Delta\Delta F = 0$ in which the Airy's function F has the property

$$\sigma_x = \frac{\partial^2 F}{\partial y^2} \quad \sigma_y = \frac{\partial^2 F}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 F}{\partial x \partial y} \qquad (1), (2), (3)$$

it is necessary to know F and its normal derivative $\frac{\partial F}{\partial n}$ on the slab boundaries.

Refer to the slab in the Figure 2 subjected on the interior boundary to a planar forces system p, situated in the slab middle plane. Consider O an arbitrary origin and BB_1 an incomplete section going to slab current point B_1. The forces resultant (external and internal) on the distance OAB_1 has the components V and H. The moment of these components in relation to B_1 is written:

$$M_{B_1} = H_y - V_x \tag{a}$$

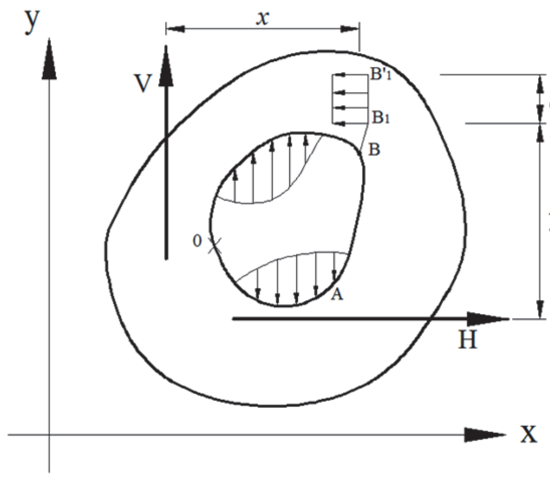


Fig. 2

Let's prolong the section form B_1 to B_1' situated at elementary distance dy from B_1. The moment in relation to B_1' forces applied on the length $OABB_1$' becomes:

$$M_{B1}^{'} = H(y + \mathrm{d}y) - V_x + \sigma_x \frac{\mathrm{d}y^2}{2} \tag{b}$$

Neglecting the last term, from the relationships (a), (b) it results:

$$M_{B_1}^{'} - M_{B_1} = H\mathrm{d}y$$

which leads to

$$H = \frac{\partial M}{\partial y} \tag{c}$$

At the same time the product $\sigma_x dy$ represents the forces elementary variation H with respect to y:

$$\sigma_x \mathrm{d}y = \frac{\partial H}{\partial y}\mathrm{d}y \tag{d}$$

From (c) and (d) relationships we deduct:

$$\sigma_x = \frac{\partial^2 M}{\partial y^2} \tag{4}$$

In the same manner, prolonging the section from B_1 to a point situated at elementary distance dx at B_1, we obtain:

$$\sigma_y = \frac{\partial^2 M}{\partial x^2} \tag{5}$$

The relationships (4), (5) having the identic form with (1), (2), it result that the stress function F in a point of plane slab, represents the moment with respect to that point of the forces (internal and external) applied on a distance which begins from an arbitrary origin and it is continued until the respective point F differs from Mby a first order polynom which may be neglected because it does not generate stresses.

If in boundary conditions

$$p_x = \sigma_x I_x + \tau_{yx} I_y \qquad p_y = \tau_{xy} I_x + \sigma_y I_y \tag{e)(f}$$

we introduce the relationships (1), (2), (3) and take into account that $I_x = \frac{\mathrm{d}y}{\mathrm{d}s}$ and $I_y = -\frac{\mathrm{d}x}{\mathrm{d}s}$ it results:

$$p_x = \frac{\partial^2 F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial^2 F}{\partial x \partial y}\frac{\mathrm{d}x}{\mathrm{d}s} = \frac{\partial}{\partial s}\left(\frac{\partial F}{\partial y}\right)$$

$$p_y = \frac{-\partial^2 F}{\partial x \partial y}\frac{\mathrm{d}y}{\mathrm{d}s} - \frac{\partial^2 F}{\partial x^2}\frac{\mathrm{d}x}{\mathrm{d}s} = -\frac{\partial}{\partial s}\left(\frac{\partial F}{\partial x}\right)$$

Integrating these expressions on the length OAB we get

$$\frac{\partial F}{\partial x} = -R_y, \frac{\partial F}{\partial y} = R_x \tag{6)(7}$$

R_y and R_x being the forces resultant components on the axis y, respectively x, concerning this distance.

Orientating the reference system after the normal n and tangent t at the boundary, the relationships (6), (7) become:

$$\frac{\partial F}{\partial n} = N \qquad \frac{\partial F}{\partial t} = T \tag{8)(9}$$

which means that: the stress function derivative with respect to normal direction n in any point of distance OAB_1 is equal to the tangent component at the boundary from the same point of the forces applied from the origin O to the respective point. This component will be named „axial force”.

Similarily, from the relationship (9) it results that the stress function derivative with respect to tangent direction t, in any point of the distance OAB_1, is equal to the normal component at the boundary from the same point, of the forces applied from origin O to respective point. This component will be named „shear force”.

Thus, it is arrived at the generalization of mechanic interpretation of Airy’s function and its derivatives for double-conex domains. This result meakes possible its determination on any slab contour subjected to an equilibrium system forces.

Really, if the point B_1 from Fig. 2, is situated on one of the slab contours and considering the origin O in any point of the same contour, in the determination of F and N, there join the external forces only and consequently it is arrived at the following practical calculation rule: considering each contour as the frame axis, open in any point and subjected to the external forces, we construct the moment and axial force diagrams; these represents, in fact, the stress function and its normal derivative values, lengthways of contour considered.

4. THE STRESSES DETERMINING IN PLANE SLAB

Using this conclusions, for the plane slab shown in figura 1 we get:

- on the exterior contour $F = 0$ and $\dfrac{\partial F}{\partial n} = 0$

- on the interior contour F and $\dfrac{\partial F}{\partial n}$ values, from Fig. 3.

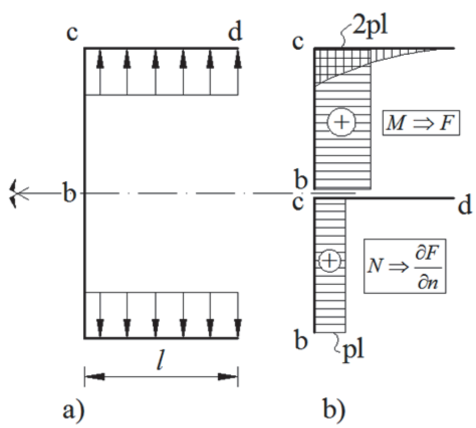


Fig. 3

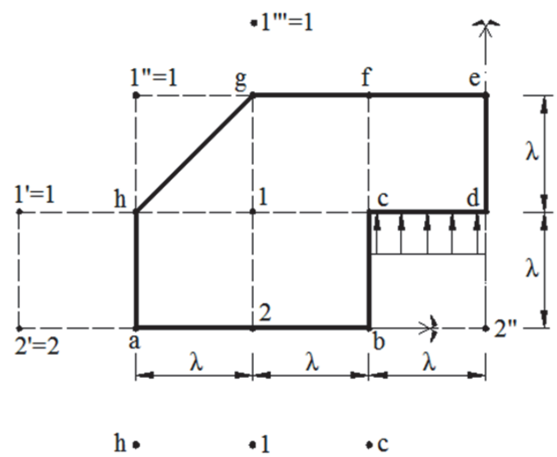


Fig. 4

The equation ΔΔF = 0 is integrated by the finite difference method. Choosing an equal step network λ and having in view the slab symmetry, this equation, written in points 1, 2 (fig. 4) becomes:

$$1 \Rightarrow 20F_1 - 8\left(F_2 + F_g + F_h + F_c\right) + 2\left(F_a + F_{1''} + F_f + F_b\right) + F_1 + F_{1'''} + F_{1'} + F_d = 0$$

(g)

$$2 \Rightarrow 20F_2 - 8\left(2F_1 + F_a + F_b\right) + 2\left(2F_h + 2F_c\right) + 2F_g + F_{2'} + F_{2''} = 0$$

But (from Fig. 3): $F_a = F_h = F_g = F_f = F_e = F_d = 0, F_c = F_b = \dfrac{pl^2}{2}$

At the same time $\dfrac{F_{1'} - F_1}{2\lambda} = N_h = 0 \Rightarrow F_{1'} = F_1$; that is: the exterior points of contour where N = 0, are symmetrical with the interior ones:

1' = 1 1" = 1 1"' = 1 2' = 2

In point *b* we have $\dfrac{F_{2''l} - F_2}{2\lambda} = pl \Rightarrow F_{2''l} = 2pl\lambda + F_2$

Similary in point d we get $\dfrac{F_{2''h} - F_e}{2\lambda} = 0 \Rightarrow F_{2''h} = F_e = 0$

Taking for $F_{2''}$ an average value, it obtains $F_{2''} = \dfrac{F_{2''l} - F_{2''h}}{2} = pl^2 + \dfrac{F_2}{2}$

With this conditions the system (g) becomes: $\begin{cases} 25F_1 - 8F_2 = 3pl^2 \\ -16F_1 + 21,5F_2 = pl^2 \end{cases}$

Its solution is: $F_1 = 0,177\,pl^2$ $F_2 = 0,178\,pl^2$

Let's determine the stresses values in any two points of slab, for example, *b* and *d*:

- point b: $\sigma_{xb} = \dfrac{2(F_c - F_b)}{\lambda^2} = 2(0,5 - 0,5)p = 0$

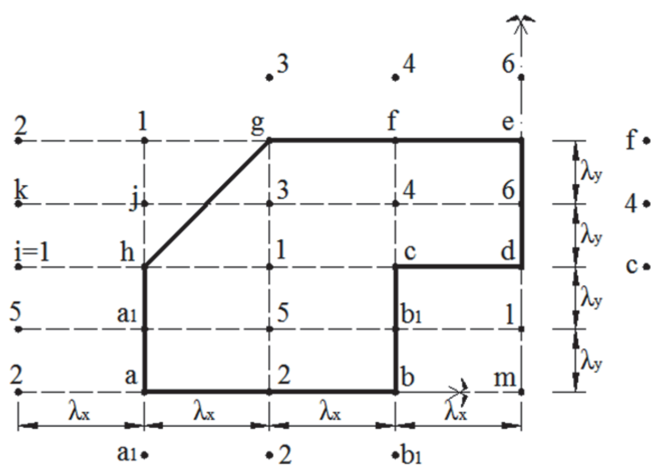
$$\sigma_{yb} = \frac{F_2 - 2F_b + F_{2''e}}{\lambda^2} = (0,178 - 2\cdot 0,5 + 2 + 0,178)p = 1,35\,p \qquad (11)$$

- point d: $\sigma_{xd} = \dfrac{F_e - 2F_d + F_{2''h}}{\lambda^2} = 0$ $\sigma_{yd} = \dfrac{2(F_c - F_b)}{\lambda^2} = 2\cdot 0,5p = p$

We can increase the results precision repeating the calculation on the network with an unequal step (fig. 5).

$$\lambda_x = \lambda \qquad \lambda_y = \lambda / 2$$

Fig. 5



In the same way we obtain:

$$1 \Rightarrow 24,5F_1 + 2F_2 - 12F_3 + 2F_4 - 12F_5 = 2pl^2$$
$$2 \Rightarrow 4F_1 + 24F_4 - 24F_5 = 0$$
$$3 \Rightarrow -12,75F_1 + 0,25F_2 + 25F_3 - 6F_4 + 2F_5 + 0,5F_6 = -pl^2$$
$$4 \Rightarrow 2,25F_1 - 6F_3 + 25,5F_4 - 6F_6 = 5pl^2$$
$$5 \Rightarrow -12F_1 - 12F_2 + 2F_3 + 25,66F_5 + 0,33F_6 = 0,666\,pl^2$$
$$6 \Rightarrow F_3 - 12F_4 + 0,66F_5 + 26,33F_6 = -3,333\,pl^2$$

The system solution is

$$F_1 = 0{,}177\,pl^2 \quad F_2 = 0{,}138\,pl^2 \quad F_3 = 0{,}081\,pl^2 \qquad (12)$$
$$F_4 = 0{,}188\,pl^2 \quad F_5 = 0{,}167\,pl^2 \quad F_6 = -0{,}048\,pl^2$$

The stresses values in the same points b and d result:

● point *b*:

$$\sigma_{xb} = 2(0{,}5 - 0{,}5)p = 0$$
$$\sigma_{yb} = (0{,}138 - 2\cdot 0{,}5 + 0{,}138 + 2)p = 1{,}28\,p \qquad (13)$$

● point *d*:

$$\sigma_{xd} = (-0{,}048 - 0{,}048)4\,p = -0{,}384\,p$$
$$\sigma_{yd} = 2\cdot 0{,}05\,p = p$$

Remark. For the considered plane slab, the stresses values in point d and b, determined by the finite element method, are:

$$\sigma_{xb} = 0 \qquad \sigma_{xd} = -0{,}45\,p \qquad (14)$$
$$\sigma_{yb} = 1{,}25\,p \qquad \sigma_{yd} = p$$

The value obtained (11) and (13) applying the plane elasticity equation are very nearly at those which result using the finit element method (14) but the calculus is much more simple (in point *d*, the stress σ_{xd}, in the first calculation stage resulted zero because the network was very roughly).

5. CONCLUSIONS

Generalizing the mechanic interpretation of Airy's function and its normal derivative at double-conex domains the plane elasticity equation application field extends in solving many engineering problems.

Using the equation $\Delta\Delta F = 0$ the calculus volume remains in easy acces limits and may be shown up by hand or with a pocket calculator.

The calculation precision is superior in comparison with other methods because the differential equations hold the privilege to render with fidelity the deformation phenomenon of strength structures.

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