PHYSICO-MATHEMATICAL MODELS OF THE VUILLEUMIER MACHINE

Adrian HOMUTESCU[1], *Vlad Mario HOMUTESCU*[2], *Condrat Adrian HOMUTESCU*[2]

[1]CONTINENTAL AUTOMOTIVE ROMÂNIA, IAŞI,
[2]TECHNICAL UNIVERSITY "GHEORGHE ASACHI", IAŞI

Rezumat. În lucrare sunt comparate mai multe modele fizico-matematice teoretice ale maşinii cu acţionare termică Vuilleumier. Se prezintă succint ipotezele fiecărui model şi legile fizice pe baza cărora poate fi simulată numeric funcţionarea maşinii şi pot fi estimate performanţele acesteia. Comparaţia între rezultatele simulării numerice a unei maşini particulare evidenţiază atât faptul că rezultatele diferitelor modele teoretice sunt relativ asemănătoare, precum şi motivele acestei asemănări.

Cuvinte cheie: acţionare termică, maşină Vuilleumier, model fizico-matematic teoretic izotermic sau semi-adiabatic

Abstract. Several theoretical physico-mathematical models of the thermally acted Vuilleumier machine are compared. The hypotheses on which each model relies together with the physical laws based on which numerical simulations of the machine can be performed and machine performances can be estimated are briefly presented. For a particular Vuilleumier machine, a good similarity is obtained by comparing the results of the numerical simulation with the results obtained by different theoretical models; the causes of this similarity are studied.

Keywords: thermally acted, Vuilleumier machine, theoretical physico-mathematical isothermal or semi-adiabatic model

1. INTRODUCTION

Reasonable use of energy in all its aspects along with the reduction of pollution are conditions without which today's sustainable development of mankind seems unconceivable. In this regard, together with the classical thermal machines, other less known and used thermal machines are more and more studied presently. These "unconventional" machines usually have several advantages that qualify them for niche utilizations. Such a technical solution is the Vuilleumier machine, technical system that can be used as heat pump, as refrigerator or as cryogenerator.

The first Vuilleumier machine was described in 1918 by engineer Rudolph Vuilleumier, who got US patent no. 1.275.507, patent named "Method and Apparatus for Inducing Heat Changes" [8].

Rudolph Vuilleumier was born in 1870 in the city of Basel in Switzerland. He emigrated to the United States of America. He worked as chief engineer at Safety Car Heating and Lighting Co. in New York. In the last 20 years of his life he lived at New Rochelle, north of New York, address mentioned on the 1918 patent. He died of pneumonia on May 12th 1920 at New Rochelle, not having the chance to take advantage of his invention [10].

A Vuilleumier machine [1], [6], [9] is a thermally acted machine inside which a constant amount of gas evolves inside an almost constant total volume. The gas lies inside several heat exchangers and four variable volume chambers placed (most often) inside two cylinders, each cylinder being fit with its own displacer piston. There are three levels of temperature inside the machine. The refrigerating effect is acquired by expanding the gas inside a low temperature chamber. Pressure variation inside the machine is acquired by heating the agent inside a high temperature chamber and by cooling the agent inside two intermediate temperature chambers.

2. VUILLEUMIER MACHINE CONSTRUCTION AND FUNCTIONING

Accordingly to the schematic diagram in fig. 1 [1], [6], a Vuilleumier machine is comprised of a cold cylinder *1* and a hot cylinder *15* inside which the cold displacer *3* and the hot displacer *13* work. The cold cylinder and displacer share a diameter inferior to the one shared by the hot cylinder and displacer. A drive comprised of crankshaft *17* and rods *16* and *18* provide movement for the displacers. The cold displacer splits the space inside its cylinder in two chambers: a low temperature one *4* and an intermediate temperature one *2*. Inside the hot cylinder the hot displacer delimitates a high temperature chamber *12* and an intermediate temperature chamber *14*. Each cylinder is fit with its own heat exchanger set. The cold cylinder has a low temperature heater *5*, a low temperature regenerator 6 and an intermediate temperature cooler *7*. The hot cylinder is fit with an intermediate temperature cooler *9*, a hot temperature regenerator *10* and a high temperature heater *11*. The intermediate temperature cooling chambers are connected through pipe *8*.

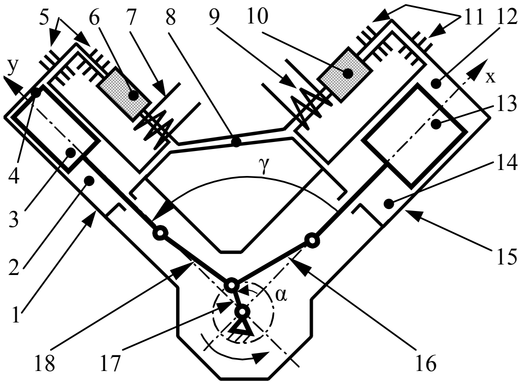

Fig. 1. Vuilleumier machine.

The phase angle between the volume variation laws ensures the presence of the working agent inside the chambers so that on thermodynamic cycle level the low temperature source and the high temperature source give heat to the agent, and the agent rejects heat to the intermediate temperature source.

The agent absorbs heat from the nearby heat sources when the pressure lowers and rejects heat to the sources when the pressure inside the machine rises; at thermodynamic cycle level, the machine absorbs heat from the low temperature source, rejecting it to the intermediate temperature heat source. In order to do that, the machine cyclically absorbs heat from the high temperature source (heat also rejected to the intermediate temperature source).

3. THEORETICAL PHYSICO-MATHEMATICAL MODELS

The physico-mathematical models and the calculation models that permit to estimate the performances of the thermally acted Vuilleumier machines can be developed through the same theoretical bases as in the case of Stirling machines (due to the constructive resemblance with that machines – the construction of both machine classes using displacers, heaters, regenerators and coolers).

The physico-mathematical models for Vuilleumier machines can be classified by means of considerring the heat transfer. This criterion evidentiates:

- isothermal models, characterized by infinite coefficients of heat transfer, in all machine chambers taking place only isothermal processes;
- adiabatic models, characterized by null coefficients of heat transfer inside variable volume chambers; here are the semi-adiabatic models also included;
- models with non-isothermal heat transfer, characterized by coeficients of heat transfer taking finite values, different from zero.

Another classification considers as main criterion the means of taking into account the energy losses inside the machine, and distinguishes models of three types - I, II and III:

- type I - does not take into account the energy losses;
- type II - takes into account the energy losses, considerring that each individual loss is produced by an independent cause. Because with this hypothesis losses do not interact with each other, these models are also named “decoupled”;
- type III - takes into account the losses and their interaction.

From the point of view of the precision of the relations used for calculations, there are:

- exact methods, that, inside the accepted hypotheses, lead to exact algebraic relations for the instantaneous pressure and for the heats exchanged during a cycle;
- approximative methods, that obtain results after numerical calculations (usually iterative, by solving a system of differential equations).

The name of “theoretical models” was used in this paper for describing the isothermal and adiabatic models without energy losses. These models appreciate the Vuilleumier machine performances using a small set of initial data, the information about the particular geometry of the machine being confined to the heat exchanger volumes and to the dimensions imposed by the chosen drive. The name “theoretical models” is completely covered by the name “type I model”, but is more suggestive.

4. ISOTHERMAL PHYSICO-MATHEMATICAL MODEL [5]

The isothermal physico-mathematical model is based on the following hypotheses:

- the working agent is the ideal gas;
- the gas amount evolving inside the machine is constant;
- at thermodynamic level all cycle functional processes are time independent;
- the metallic parts of the machine (other than heat exchangers and cylinder walls) do not exchange heat either among them or with the exterior;
- the processes inside heat regenerators are ideal ones (regeneration efficiencies are 100%);
- the agent temperature inside the low temperature chamber is equal to the one inside the low temperature heater, the one of the outer heating agent, the one of the cylinder walls and the one of the cold displacer frontal surface;
- the agent temperature inside the high temperature chamber is equal to the one inside the high temperature heater, the one of the outer heating agent, the one of the cylinder walls and the one of the hot displacer frontal surface;
- the agent temperature inside the intermediate temperature chambers is equal to the one inside the coolers, the one of the outer cooling agent, the one of the cylinder walls next to those respective chambers and the one of the stems and of the displacer bottoms;
- the instantaneous pressure is identical in all the spaces occupied by the agent, its value varying along the cycle;
- the volume variation law for the chambers inside the machine cylinders is known.

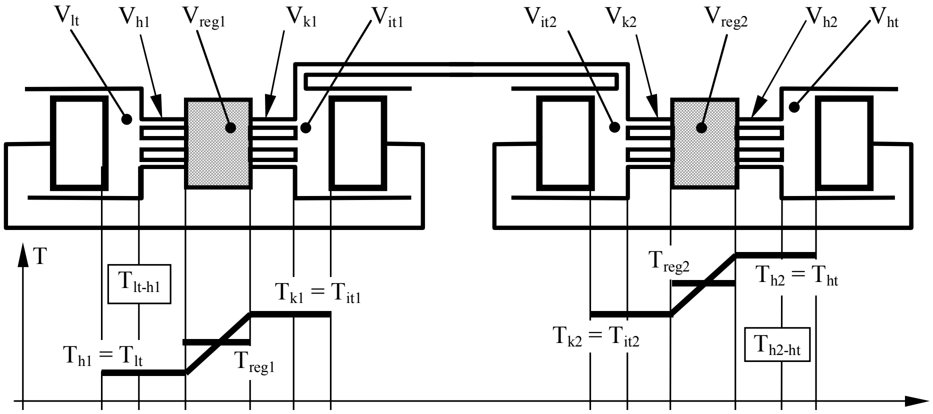

Fig. 2. Isothermal model of the Vuilleumier machine.

The hypotheses implying the temperatures inside the Vuilleumier machine show that inside all chambers isothermal processes take place only, thus confirming the described physico-mathematical model the denomination of isothermal model. To outline the isothermal character of the physico-mathematical model analyzed here, on fig. 2 the machine chambers are separate and placed in row. This presentation required the halving of each displacer. Each variable volume chamber is assigned half a displacer. The mechanical linkage between the displacer halves was symbolically drawn through bars exterior to the cylinder.

The following subscripts for dimensions inside machine chambers (volume V, temperature T, mass m) were used: h = heater; reg = regenerator; k = cooler; 1 = cold displacer; 2 - hot displacer; ht = high temperature; lt = low temperature; it = intermediate temperature. Subscript “i” represents a generic chamber and takes values from “lt” to “ht”.

The pressure variation law $p(\alpha)$ was determined from the equation of conservation for the total gas mass inside the machine, where the masses m_i are calculated for each chamber from the equation of state.

The pressure acquires the expression:

$$p(\alpha) = \frac{m\,R}{\sum_i \frac{V_i(\alpha)}{T_i}} \; . \tag{1}$$

Temperature values T_{lt}, T_{it}, T_{ht} are imposed by user and the chamber volumes are imposed by the geometry chosen for the machine. Regenerator mean temperatures are calculated as logarithmic mean between the neighbouring chambers’ temperatures.

Accordingly to the first law of thermodynamics applied to the agent undergoing an open cycle inside an isothermal chamber, the heat exchanged during a complete cycle is equal to the work exchanged, being calculated with the relations:

$$Q_i = \int_0^{2\pi} p(\alpha)\left[\frac{dV_i(\alpha)}{d\alpha}\right] d\alpha \, . \tag{2}$$

The useful effect of the Vuilleumier refrigerating machine is represented by heat Q_{lt} extracted from the low temperature heat source and the useful effect of the heat pump is heat (Q_{it1} +Q_{it2}) transmitted to the user at an intermediate temperature T_{it}.

The Vuilleumier refrigerating machine is characterized by the coefficient of performance

$$\varepsilon_r = \frac{|Q_{lt}|}{Q_{ht}} \tag{3}$$

and the heat pump by the coefficient of performance

$$\varepsilon_{hp} = \frac{|Q_{it1} + Q_{it2}|}{Q_{ht}} \, . \tag{4}$$

As a consequence of the presence of the displacer stems, the machine exchanges with the environment an amount of work per cycle having the expression

$$L = Q_{lt} + Q_{it1} + Q_{it2} + Q_{ht} \, . \tag{5}$$

5. SIMPLIFIED ISOTHERMAL PHYSICO-MATHEMATICAL MODEL [3]

The isothermal model can be simplified, by altering some of the hypotheses, to the point when the integrals from eq. (2) can be analytically integrated. So, exact algebraic relations (calculation formulae) for the heats exchanged and for the coefficients of performance can be obtained. Such a simplified method was used for the first time by Gustav Schmidt in 1871 for a Stirling engine [6]. The model operates with a machine inside which the movement of the displacers is harmonic. The volumes of the heaters and coolers are neglected (fig. 3). The agent temperature inside the regenerator is deemed constant, and is calculated as arithmetic mean. The strict use of the conditions imposed by Gustav Schmidt assumes that no dead spaces are considered and the diameter of the displacer stems are null.

The COP’s and the heat exchanged by a Vuilleumier machine having the composition shown on fig. 3 and respecting the hypotheses of the simplified isothermal model (Schmidt-type) can be calculated by numerical integration, in the same manner as at the isothermal model, or can be calculated with the specific formulae [3].

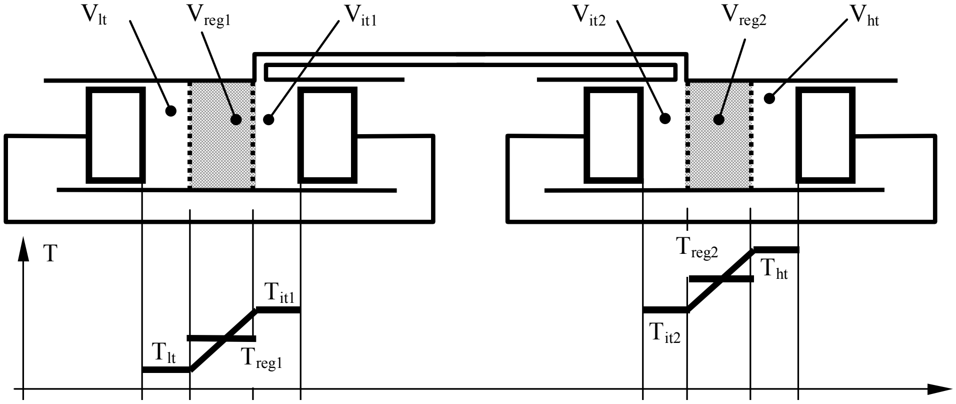

Fig. 3. Simplified isothermal model of the Vuilleumier machine (Schmidt-type).

6. SEMI-ADIABATIC MODEL FOR THE VUILLEUMIER MACHINE [2]

The name semi-adiabatic is used for describing an adiabatic model where only some of the chambers with variable volume are considered to be adiabatic. So, the name can be applied to more than one distinct model. The main hypothesis of the chosen semi-adiabatic model is that inside the low temperature chamber *4* (on fig. 1) and inside the high temperature chamber *12* adiabatic processes take place; so, the temperature inside these chambers vary cyclically. The other hypotheses are taken from the isothermal model "as is" or modified in concordance with the main assumption.

The model uses the differential equation of mass conservation for the working agent, the equation of state applied to the heat exchangers and to the intermediate temperature chambers and the differential law of conservation of energy written for the adiabatic chambers [7].

Accordingly to the adopted hypotheses, inside the low temperature chamber and inside the high temperature chamber the gas exchanges work with the surrounding environment (through piston movement) and enthalpy with the neighbouring chambers (through agent's entering the chamber from the neighbouring heater or leaving it toward the heater). The internal energy of the gas inside the adiabatic chamber changes, as a consequence of mass and temperature variations. Inside these two chambers the heat exchanged is zero, conforming to the adiabatic hypothesis.

The semi-adiabatic model produces five differential equations:

$$dp = \frac{-k\,p\left[\frac{dV_{lt}}{T_{lt-h1}} + \frac{dV_{ht}}{T_{h2-ht}} + \frac{dV_{it1}}{T_{it1}} + \frac{dV_{it2}}{T_{it2}}\right]}{\frac{V_{lt}}{T_{lt-h1}} + \frac{V_{ht}}{T_{h2-ht}} + k\left[\frac{V_{it1}}{T_{it1}} + \frac{V_{it2}}{T_{it2}} + \left(\frac{V_{h1}}{T_{h1}} + \frac{V_{reg1}}{T_{reg1}} + \frac{V_{c1}}{T_{k1}} + \frac{V_{c2}}{T_{k2}} + \frac{V_{reg2}}{T_{reg2}} + \frac{V_{h2}}{T_{h2}}\right)\right]}; \qquad (6)$$

$$dm_{lt} = \frac{1}{R\,T_{lt-h1}}\left[p\,dV_{lt} + \frac{V_{lt}}{k}dp\right]; \qquad (7)$$

$$dm_{ht} = \frac{1}{R\,T_{h2-ht}}\left[p\,dV_{ht} + \frac{V_{ht}}{k}dp\right]; \qquad (8)$$

$$dT_{lt} = T_{lt}\left(\frac{dp}{p} + \frac{dV_{lt}}{V_{lt}} - \frac{dm_{lt}}{m_{lt}}\right); \qquad (9)$$

$$dT_{ht} = T_{ht}\left(\frac{dp}{p} + \frac{dV_{ht}}{V_{ht}} - \frac{dm_{ht}}{m_{ht}}\right). \qquad (10)$$

The composed subscripts lt-h1 şi h2-ht refer to the dimensions describing the separating sections between the low and high temperature chambers and its adjacent heaters.

Equations (6), (7), (8), (9) and (10) form the system of differential equations of the semi-adiabatic physico-mathematical model of the Vuilleumier machine. The unknown functions are the pressure p, the masses m_{lt} and m_{ht} inside the low and high temperature chambers and the temperatures T_{lt} and T_{ht} in the same chambers. The system is nonlinear, because there are several terms in the differential equations that have an order higher than one. The system has variable coefficients and the conditional temperatures T_{lt-h1} and T_{h2-ht} of the agent that passes through the surfaces lt-h1 and h2-ht depend on the sense of the gas flow. The conditional temperatures take the expressions:

$$\begin{aligned} &T_{lt-h1} = T_{lt} \quad \text{if} \quad m_{lt-h1} > 0 \quad (\text{or} \quad dm_{lt} < 0); \\ &T_{lt-h1} = T_{h1} \quad \text{if} \quad m_{lt-h1} < 0 \quad (\text{or} \quad dm_{lt} > 0); \\ &T_{h2-ht} = T_{h2} \quad \text{if} \quad m_{h2-ht} > 0 \quad (\text{or} \quad dm_{ht} > 0); \\ &T_{h2-ht} = T_{ht} \quad \text{if} \quad m_{h2-ht} < 0 \quad (\text{or} \quad dm_{ht} < 0). \end{aligned} \qquad (11)$$

The system can be solved only by numerical integration. If the values of the unknown functions are adopted for a certain point in time, the problem is an initial value one and the numerical solution can be found with a Runge-Kutta method. The solution is obtained after several iterations, each of them using the previous one's results as initial values and thus getting closer to the result as the analysis goes on.

The useful effect of the Vuilleumier refrigerating machine is represented by heat Q_{h1} extracted from the low temperature heat source (inside heater h1) and the useful effect of the heat pump is heat (Q_{it1} +Q_{it2}) transmitted to the user at an intermediate temperature of T_{it}.

The Vuilleumier machine performances can be calculated with the general relations (3), (4) and (5) given at the isothermal model.

7. ADIABATIC MODEL FOR THE VUILLEUMIER MACHINE [4]

The adiabatic model assumes that in all variable volume chambers (lt, it1, it2 and ht) adiabatic processes take place only. The model produces a system of nine differential equations, closely resembling the equations obtained for the semi-adiabatic model. The unknown functions are the pressure inside the machine, the masses and temperatures inside the adiabatic chambers.

8. COMPARATION BETWEEN PHYSICO-MATHEMATICAL MODELS. NUMERICAL EXAMPLE

The above physico-mathematical models were applied to a Vuilleumier machine featuring the following dimensions: D_l= 0.1 m; $d_1 = d_2$ = 0.015 m; D_2 = 0.12 m; $r_l = r_2$ = = 0.05 m; $l_l = l_2$ = 0.2 m; $f_{TDPl} = f_{BDPl} = f_{TDP2} = f_{BDP2}$ = = 0.001 m; $V_{hl} = V_{h2} = V_{kl} = V_{k2} = 0.05\ V_{Sht}$; $V_{regl} = V_{reg2}$ = = 1.2 V_{Sht}, $\gamma = \pi/2$, where V_{Sht} = maximum volume of the high temperature chamber, D = cylinder diameter, d = stem diameter, r = crankshaft radius, l = rod length, f = dead space length, γ = angle between cylinder axes.

The machine works with a total mass of hydrogen m = 0.0207 kg (corresponding to a pressure of 50 bar inside the machine, at an ambient temperature of 15 ºC; R_{H2} = 4124.48 J/(kg K)) between temperatures $T_{ht} = T_{h2}$ = = 923 K; $T_{kl} = T_{k2} = T_{itl} = T_{it2}$ = 343 K and $T_{lt} = T_{hl}$ = = 278 K (for the semi-adiabatic model, T_{ht} and T_{lt} vary according to the adiabatic behaviour of these chambers).

The Vuilleumier machine functioning was simulated with the isothermal, simplified isothermal and semi-adiabatic models. The results of the numerical simulation are displayed in fig. 4, fig. 5, fig. 6 and fig. 7, as well as inside table 1.

Table 1

Calculated results

Model	Q_{lt}/Q_{h1}	Q_{ht}/Q_{h2}	Q_{it1}	Q_{it2}	ε_{hp}
	[J/cycle]				-
isothermal simplified	805.3	299.6	-805.3	-299.6	3.69
isothermal	718.0	300.4	-701.9	-295.7	3.32
semi-adiabatic	749.4	350.3	-732.4	-344.7	3.07

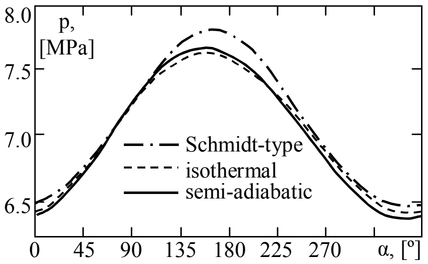

Fig. 4. Pressure variation inside Vuilleumier machine.

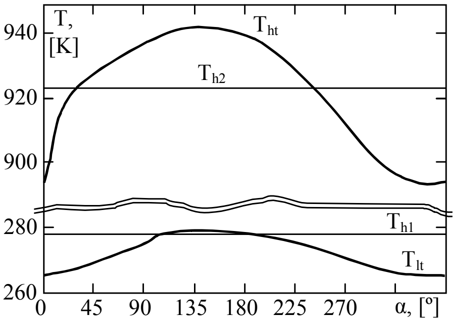

Fig. 5. Temperature variation inside the low and high temperature adiabatic chambers.

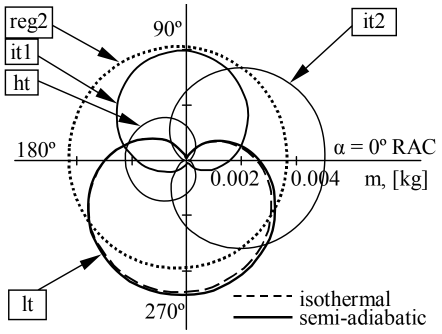

Fig. 6. Mass variation inside the variable volume chambers.

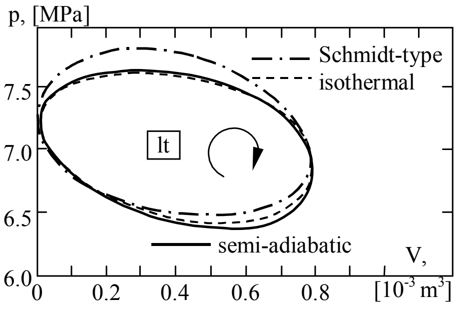

Fig. 7. Pressure-volume indicator diagram inside low temperature chamber.

9. CONCLUSIONS

The physico-mathematical models proposed for the numerical simulation of the Vuilleumier machine functioning allow for providing information on the possible performance the machine is capable of. Inside a real machine the heat exchanges do not take place isothermally, the heat regeneration is not ideal and the agent flow through the heat exchangers occurs with friction, all these facts lowering the COP's under the calculated ones.

The pressure variation inside the machine, calculated with the semi-adiabatic model, is very close to the ones calculated with the isothermal models, because a large quantity of the working agent is placed inside chambers considered to be isothermal in all models, especially inside regenerators.

At the semi-adiabatic model the temperature inside the low temperature chamber is – for the most part of the cycle – below the neighbouring isothermal heater temperature. The mean temperature inside this adiabatic chamber is below the cold heater temperature.

All models stress the heat amounts exchanged inside the machine chambers. Accordingly to the adopted hypotheses, at the isothermal models the heaters and coolers stand for dead spaces attached to the neighbouring variable volume chambers. At the semi-adiabatic model, because the heaters are adjacent to the adiabatic chambers, they cyclically exchange nonzero heats.

The heat amounts exchanged with the heat sources calculated with the semi-adiabatic model are larger than the corresponding ones calculated with the isothermal model. The coefficient of performance (COP) is smaller than the isothermal one.

The semi-adiabatic model allows for a rapid analysis of the influence some constructive and functional factors have (more than the isothermal models can support) as well as for comparing different machines.

The energetic balance per cycle for the Vuilleumier machine, written in expression (5), shows that the machine produces a small amount of work also, as result of the piston stem presence. Inside the real machine, this amount is insufficient to compensate for the friction losses.

Results superior to other models obtained with the simplified isothermal model (Schmidt-type) are explained by the reduced total volume of the machine and by the use of null diameter of the stems.

The close similarity of the results given by the theoretical models of the Vuilleumier machine shows that for the study of the calitative analysis of the influences that several constructive and functional factors exert over machine performances it is sufficient to use the least complicated model describing the studied phenomenon.

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